A Globally Stable Adaptive Congestion Control Scheme for Internet-Style Networks with Delay

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Abstract— **In this paper, we develop, analyze and implement a congestion control scheme in a noncooperative game framework, where each user's cost function is composed of a pricing function proportional to the queueing delay experienced by the user, and a fairly general utility function which captures the user demand for bandwidth. Using a network model based on fluid approximations and through a realistic modeling of queues, we establish the existence of a unique equilibrium as well as its global asymptotic stability for a general network topology, where boundary effects are also taken into account. We also provide sufficient conditions for system stability when there is a bottleneck link shared by multiple users experiencing non-negligible communication delays. In addition, we study an adaptive pricing scheme using hybrid systems concepts. Based on these theoretical foundations, we implement a window-based, end-to-end congestion control scheme, and simulate it in *ns-2* network simulator on various network topologies with sizable propagation delays.**

Methods Keywords: Control theory, Mathematical programming/optimization, Simulations, Economics.

Index Terms— **Congestion control; Internet; noncooperative games; stability.**

I. Introduction

A. Background and Related Work

Congestion control is one of the critical issues that lie at the heart of efficient operation of the Internet. The goal of congestion control can be summarized as regulation of the source rates in a decentralized and distributed fashion such that the available bandwidths on the links are used most efficiently while minimizing loss of packets due to buffer overflows at the routers. In recent years, the congestion control problem and modeling of the Internet have caught the attention of the research community. After the introduction of the congestion control algorithm for transfer control protocol (TCP) [1], focus has been on the modeling and analysis of such algorithms. Based on an earlier work by Kelly [2], Kelly, Mauloo, and Tan [3] have presented the first comprehensive mathematical model, and posed the underlying resource allocation in congestion control as an optimization problem. The primal and dual algorithms they have introduced are based on user utility and link pricing (explicit feedback) functions, where the sum of user utilities are maximized within the capacity (bandwidth) constraints of the links. Furthermore, they have established the global stability of these algorithms in the no-delay case as well as under small perturbances. They have also introduced the concept of proportional fairness, which is a relaxed version of min-max fairness [4], as a resource allocation criterion among users.

Subsequent studies [5]–[9] have investigated variations and generalizations of the distributed congestion control framework of [2], [3]. Low and Lapsley [7] have analyzed the convergence of distributed synchronous and asynchronous discrete algorithms, which solve a similar optimization problem. Mo and Walrand [5] have generalized the proportional fairness, and have proposed a fair end-to-end window based congestion control scheme, which is similar to primal algorithm. The main difference of this window based algorithm from the primal algorithm is that it does not need feedback from the routers due to usage of queuing delay as feedback. La and Anantharam [6] have considered a system model similar to proposed in [5] with a window-based control scheme and static modeling of link buffers. They have investigated convergence properties of the proposed charge-sensitive congestion control scheme, which utilizes a static pricing scheme based on link queueing delays. In addition, they have established stability of the algorithm at a single bottleneck node. Kunniyur and Srikant [9] have examined the question of how to provide the congestion feedback from the network to the user. They have proposed an explicit congestion notification (ECN) marking scheme combined with dynamic adaptive virtual queues, and have shown using a time-scale decomposition that the system is semi-globally stable in the no-delay case.

Stability properties of primal, dual, and similar algorithms have recently been investigated in the presence of non-negligible delays [10]–[12]. Johari and Tan [12] have analyzed the local stability of a delayed system where the end user implements the primal algorithm. They have considered a single link accessed by a single user, as well as its multiple user extension under the assumption of symmetric delays. In both cases, they have provided sufficient conditions for local stability of the underlying system of equations. Massoulie [11] has extended these local stability results to general network topologies and heterogeneous delays. In another study, Vinnicombe [10] has also provided sufficient conditions for local stability of a user source law which is a generalization of the same algorithm. Elwalid [13] has considered stability of a linear class of algorithms where the source rate varies in proportion to the difference between the buffer content and the target value. Deb and Srikant [8], on the other hand, have focused on the case of single user and a single resource and investigated sufficient conditions for global stability of various nonlinear congestion control schemes under fixed information

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delays. Liu, Başar, and Srikant [14] have extended the framework of [2], [3] by introducing a primal-dual algorithm which has dynamic adaptations at both ends (users and links), and have given a condition for its local stability under delay using the generalized Nyquist criterion. Wen and Arcak [15] have used the passivity framework to unify some of the stability results on primal and dual algorithms without delay, have introduced and analyzed a larger class of such algorithms for stability, and have shown robustness to variations due to delay.

B. Overview and Contribution of the Study

In developing pricing and congestion control mechanisms for the Internet, game theory provides a natural framework. Users on the network can be modeled as players in a congestion control game where they choose their strategies or in this case flow rates. Players are noncooperative in terms of their demands for network resources, and have no specific information on other users' strategies. A user's demand or utility for bandwidth is captured in a utility function which may not be bounded. To compensate for this, one can devise a pricing function, proportional to the bandwidth usage of a user, as a disincentive to him to have excessive demand for bandwidth. This way, the network resources are preserved, and an incentive is provided for the user to implement end-to-end congestion control [16]. A useful concept in such a noncooperative congestion control game is that of Nash equilibrium [17] where each player minimizes his/her own cost (or maximize payoff) given all other players' strategies. This is already a well-acknowledged field of study for communication networks, and there is a rich literature on game theoretic analysis of flow control problems utilizing both cooperative [18] and noncooperative [19]–[25] frameworks.

Although the game theoretic approach provides a suitable framework for formulating and studying congestion and flow control problems in general networks, there are some inherent restrictions on implementable cost functions in the case of Internet-style networks. For example, the current structure of the Internet makes it difficult, if not impossible, for users to obtain detailed real time information on the state of the network and on other users. Therefore, users are bound to use indirect aggregate metrics that are available to them, such as packet drop rate or variations in the average round trip time (RTT) of packets in order to infer the current situation in the network. Packet drops, for example, are currently used by most widely deployed versions of TCP as an indication of congestion. In this paper, however, we propose and analyze a pricing and congestion control scheme based on variations in the RTT a user experiences. A similar approach has been suggested in a version of TCP, known as TCP Vegas [26]. Although TCP Vegas is more efficient than a widely used version of TCP, TCP Reno [27], the suggested improvements are empirical and based on experimental studies.

The noncooperative congestion control game introduced in this paper is characterized by a cost function for each user that is defined as the difference of pricing and utility functions. The pricing function is proportional to the queueing delay experienced by the user, whereas the utility function that quantifies the user demand for bandwidth belongs to a broad class of strictly increasing and strictly concave functions. The framework in this paper differs from our related work in [22]–[24] in the sense that here we ignore the effect of a user's flow on congestion feedback. This assumption holds especially if the number of users is large. Therefore, unlike our other studies, the unique equilibrium point of the system here is only an approximation to 'Nash' equilibrium, which leads to an alternate interpretation of this study within the well known framework of [2], [3]. Here, we introduce a model with both user and pricing (queueing) dynamics within the so called *primal-dual* framework [4] for solving the resource allocation problem of [3]. However, we focus here strictly on end-to-end congestion control where no modification to routers are needed in implementation. Hence, our approach should not be confused with existing active queue management (AQM) schemes.

We show the existence of a unique 'Nash' or system equilibrium through a network model based on fluid approximations, and a realistic queueing model. Furthermore, we establish the global stability of the equilibrium under a general network topology. We also investigate global stability of the system in a network with non-negligible propagation delays, and provide sufficient conditions for global stability in the case of a bottleneck node with multiple users. In addition, we study an adaptive pricing scheme used to adjust the pricing parameter dynamically, and make use of hybrid (switched) system concepts for its analysis. Our focus here is on the design of a congestion control scheme which does not require any modification to the network core and routers. However, if a virtual queuing scheme is implemented in network buffers as described in [9] then marked packets may be utilized in the adaptive pricing algorithm to prevent packet losses altogether. Based on the theoretical foundations developed, we design a window-based, end-to-end congestion control scheme for Internet-style networks, which is TCP-friendly [28]. This congestion control scheme is then simulated in Network Simulator 2 (*ns-2*) over Internet protocol (IP) for various network topologies. An earlier version of this paper without the adaptive pricing scheme and without a rigorous analysis of stability with boundary effects, can be found in [25].

Due to the large body of work in this area, it is necessary to compare this study with earlier contributions. The studies by Mo and Walrand [5], and La and Ananatharam [29] also make use of an approach similar to the one in this paper. However, the former study employs only a small set of utility functions in describing user demands, while the latter does not take into account queue dynamics or the effect of boundaries and propagation delays on stability of the system. The "stability under delay" results in most of the earlier work are restricted either to specific cases such as symmetric delays [12], linear class of algorithms [13], single-link single-user [8], or to local stability [10], [11], [14] only. On the other hand, global stability results in this study capture the fairly general case of multiple users with fixed heterogeneous delays to the bottleneck link, under the realistic primal-dual system model. Other main contributions here are the rigorous boundary analysis and the utilization of the hybrid system framework and sliding mode behavior for adaptive online adjustment of the cost parameters.

The rest of the paper is organized as follows: The underlying network model and cost function are given in the next section. In Section III, the existence of a unique equilibrium and global stability of the system under a general network topology are established. Section IV generalizes the stability analysis of Section III to the case with delay, with a single bottleneck link. We consider in Section V an adaptive pricing scheme, and analyze it within a hybrid system framework. In Section VI we provide a realistic implementation of the congestion control scheme for IP networks. Section VII includes simulation results, and is followed by the concluding remarks of Section VIII.

II. The Model

A. The Network Model

We consider a general network model based on fluid approximations. Fluid models are widely used in addressing a variety of network control problems, such as congestion control [5], [20], [22], routing [20], [21], and pricing [3], [18], [30]. The topology of the network is characterized by a set of nodes $\mathcal{N} = \{1, \ldots, N\}$, and a set of links $\mathcal{L} = \{1, \ldots, L\}$ connecting the nodes. In this network model, we make the natural assumption of *connectivity*, and let $\mathcal{M} := \{1, \ldots, M\}$ denote the set of active users. Each link $l \in \mathcal{L}$ has a fixed capacity $C_l > 0$, and an associated buffer size $b_l \geq 0$. For simplicity, each user is associated with a (unique) connection. Hence, the i^{th} ($i \in \mathcal{M}$) user corresponds to a unique connection between the source and destination nodes, $s_i, de_i \in \mathcal{N}$, and we denote the corresponding route (path), which is a subset of $\mathcal{L}$, by R_i. The nonnegative flow, x_i, sent by the i^{th} user over this path R_i satisfies the bounds $0 \leq x_i \leq x_{i,max}$. The upper bound, $x_{i,max}$, on the i^{th} user's flow rate may be a user specific physical limitation.

It is possible to define a routing matrix, $\mathbf{A}$, as in [3] that describes the relation between the set of routes $\mathcal{R}$ associated with the users (connections) and links $l \in \mathcal{L}$:

$$A_{l,i} = \begin{cases} 1, & \text{if source i uses link}\, l \quad\quad i \in \mathcal{M}\, and \\ 0, & \text{if source i does not use link}\, l \quad l \in \mathcal{L} \end{cases} \tag{1}$$

Using this routing matrix $\mathbf{A}$, the capacity constraints of the links are given by $\mathbf{Ax} \leq \mathbf{C}$, where $\mathbf{x}$ is the $(M \times 1)$ flow rate vector of the users and $\mathbf{C}$ is the $(L \times 1)$ link capacity vector. If the aggregate sending rate of users whose flows pass through link l exceeds the capacity, C_l, of that link then the arriving packets are queued (generally on a first-come first-serve basis) in the buffer, b_l, of the link with $b_{l,max}$ being the maximum buffer size. Furthermore, if the buffer of the link is full, incoming packets have to be dropped. Let the total flow on link l be given by $\bar{x}_l := \sum_{i:l \in R_i} x_i$. Thus, the buffer level at link l evolves in accordance with

$$\dot{b}_l(t) = \begin{cases} [\bar{x}_l - C_l]^-, & \text{if } \; b_l(t) = b_{l,max} \\ \bar{x}_l - C_l, & \text{if } \; 0 < b_l(t) < b_{l,max} \\ [\bar{x}_l - C_l]^+, & \text{if } \; b_l(t) = 0 \end{cases} \tag{2}$$

where $\dot{b}_l(t)$ denotes $(\partial b_l(t)/\partial t)$, $[.]^+$ represents the function $\max(.\,, 0)$, and $[.]^-$ represents the function $\min(.\,, 0)$. In addition, let $\dot{\mathbf{b}} := [\dot{b}_1, \ldots, \dot{b}_L]$ be the $(L \times 1)$ link buffer rate vector, and $\mathbf{O} := [O_1, \ldots, O_L]$ be defined as the $(L \times 1)$ flow loss rate vector at the links, where

$$O_l := \begin{cases} [\bar{x}_l - C_l]^+, & \text{if } \; b_l = b_{l,max} \\ 0, & \text{otherwise} \end{cases}$$

Taking the buffer dynamics and packet losses into account, we redefine the capacity constraints at the links as

$$\mathbf{Ax}(t) - \dot{\mathbf{b}}(t) - \mathbf{O}(t) \leq \mathbf{C}. \tag{3}$$

B. The Primal-Dual Algorithm

An important indication of congestion for internet-style networks is the variation in queueing delay, d, which is defined as the difference between the actual delay experienced by a packet, d^a, and the fixed propagation delay of the connection, d^p. If the incoming flow rate to a router exceeds its capacity, packets are queued (generally on a first-come first-serve basis) in the existing buffer of the router, leading to an increase in the RTT of packets. Hence, RTT on a congested path is larger than the base RTT, which is defined as the sum of propagation and processing delays on the path of a packet. The queueing delay at the l^{th} link, d_l, is a nonlinear function of the excess flow on that link, given by

$$\dot{d}_l(\mathbf{x}, t) = \begin{cases} \left[\dfrac{1}{C_l}(\bar{x}_l - C_l)\right]^-, & \text{if } d_l(t) = d_{l,max} \\ \dfrac{1}{C_l}(\bar{x}_l - C_l), & \text{if } 0 < d_l(t) < d_{l,max} \\ \left[\dfrac{1}{C_l}(\bar{x}_l - C_l)\right]^+, & \text{if } d_l(t) = 0 \end{cases} \tag{4}$$

in accordance with the buffer model described in (2), with $d_{l,max} := b_{l,max}/C_l$ being the maximum possible queueing delay. Thus, the total queueing delay, D_i, a user experiences is the sum of queueing delays on its path, namely $D_i(\mathbf{x}, t) = \sum_{l \in R_i} d_l(\mathbf{x}, t)$, $i \in \mathcal{M}$, which we henceforth write as $D_i(t),\ i \in \mathcal{M}$.

Let us define a cost function for each user as the difference between pricing and utility functions. The pricing function of the i^{th} user is linear in x_i for each fixed total queueing delay D_i of the user, and is linear in D_i with x_i fixed (hence it is a bilinear function of x_i and D_i). The utility function $U_i(x_i)$ is assumed to be strictly increasing, differentiable, and strictly concave; it basically describes the user's demand for bandwidth. Accordingly, we make use of variations in RTT to devise a congestion control and pricing scheme. The cost (objective) function for the i^{th} user at time t is thus given by

$$J_i(\mathbf{x}, t) = \alpha_i D_i(t)\, x_i \, - U_i(x_i) \;, \tag{5}$$

which s/he wishes to minimize. In accordance with this objective, we consider a simple dynamic model of the network game where each user changes her/his flow rate in proportion with the gradient of her/his cost function with respect to her/his flow rate, $\dot{x}_i = -\partial J_i(\mathbf{x})/\partial x_i$. Taking into consideration also the boundary effects, the update algorithm for the i^{th} user thus is:

$$\dot{x}_i = \begin{cases} \left[\dfrac{dU_i(x_i)}{dx_i} - \alpha_i D_i(t)\right]^-, & \text{if } x_i = x_{i,max} \\ \dfrac{dU_i(x_i)}{dx_i} - \alpha_i D_i(t), & \text{if } 0 < x_i < x_{i,max} \\ \left[\dfrac{dU_i(x_i)}{dx_i} - \alpha_i D_i(t)\right]^+, & \text{if } x_i = 0 \end{cases} \quad (6)$$

where the effect of the i^{th} user on the delay, $D_i(t)$, s/he experiences is ignored. This assumption can be justified for networks with a large number of users, where the effect of each user is vanishingly small. Furthermore, from a practical point of view, it is extremely difficult if not impossible for a user to estimate her/his own effect on queueing delay.

C. Model Assumptions

We provide in this subsection a summary of the main assumptions which will be utilized in Sections III, IV, and V. These simplifying assumptions are necessary to arrive a mathematically tractable model, and most of them are shared by the majority of the literature on the subject, including the works we have cited in Section I. We note that the analytical results obtained based on these assumptions are demonstrated via realistic packet level simulations in Section VII. Now, our main assumptions are the following: (a) The network model considered is based on fluid approximations, where individual packets are replaced with flows. (b) For simplicity, each user is associated with a unique connection and a corresponding fixed route (path). The routing matrix $\mathbf{A}$ is assumed to be of full row rank as non-bottleneck links have no effect on the equilibrium point due to zero queuing delay on those links. (c) The effect of individual packet losses on the flow rates are ignored. This approximation is reasonable as one of the main goals of our congestion control scheme is to minimize or totally eliminate packet losses. (d) Information delays in Section IV are assumed to be fixed for tractability of analysis. (e) We assume first-in first-out (FIFO) finite queues (buffers) with droptail packet dropping policies. (f) The effect of a user on his/her own queueing delay is ignored, which is justified for networks with a large number of users. (g) The utility function $U_i(x_i)$ of the i^{th} user is assumed to be strictly increasing and concave in x_i.

III. Stability Analysis

In this section, we analyze the stability of the system described by (4) and (6). First, we investigate the case of a single link but multiple users. We then generalize the analysis to a general network topology with multiple links and users.

A. Stability for a Single Link with Multiple Users

For the single link case, the system dynamics become:

$$\begin{aligned} \dot{x}_i(t) &= \frac{dU_i(x_i)}{dx_i} - \alpha_i d(t), \ i = 1, \ldots, M, \\ \dot{d}(t) &= \frac{\sum_{i=1}^M x_i}{C} - 1, \end{aligned} \quad (7)$$

where $U_1(x_1), \ldots, U_M(x_M)$ are strictly concave user utility functions, and the boundary point behavior is described by (4) and (6). Let us define the corresponding constraint set Ω as

$$\Omega = \{(\mathbf{x}, d) \in \mathbb{R}^{M+1} : 0 \le x_i \le x_{i,max}, \ \forall i \in \mathcal{M} \text{ and } 0 \le d \le d_{max}\}, \quad (8)$$

where d_{max} and $x_{i,max}$ are upper-bounds on d and x_i respectively.

We investigate the existence of a unique inner equilibrium point under the assumption $x_{i,max} > C, \forall i \in \mathcal{M}$. To simplify the notation, let $p_i(x_i) := dU_i(x_i)/dx_i$, where p_i is strictly decreasing in x_i due to strict concavity of U_i. Further define $P(\alpha, d) := \sum_{i=1}^M p_i^{-1}(\alpha_i d)$ where p_i^{-1} is the inverse of the function p_i. We note that $p_i^{-1}(\alpha_i d)$ is strictly decreasing in both α_i and d, and therefore $P(\alpha, d)$ is also decreasing in the pair (α, d). Ignoring the boundary effects, the equilibrium point of (7) is given by

$$x_i^* = p_i^{-1}(\alpha_i d^*), \ \forall i \in \mathcal{M}, \quad (9)$$

where d^* is solved from

$$P(\alpha, d^*) := \sum_{i=1}^M p_i^{-1}(\alpha_i d^*) = C. \quad (10)$$

We now show that $(\mathbf{x}^*, d^*)$ is indeed an inner equilibrium solution to (7) on the set Ω for some range of α values. Let the vector α belong to the compact set $\{\alpha \in \mathbb{R}^M : 0 \le \alpha_{i,min} \le \alpha_i \le \alpha_{i,max}, \ \forall i \in \mathcal{M}\}$. Define $\epsilon > 0$ to be arbitrarily small. It follows from the definition of $P(\alpha, d)$ that $P(\alpha_{max}, \epsilon) > C$ for arbitrarily large values of $\alpha_{i,max}$ $\forall i \in \mathcal{M}$. Furthermore, given d_{max} one can find a vector α_{min} such that

$$P(\alpha_{min}, d_{max}) < C, \quad (11)$$

Using the Intermediate Value Theorem, we conclude that there exists a unique $d^* \in (d_{min}, d_{max})$ such that $P(\alpha, d^*) = C$. In addition, (9) yields a unique $\mathbf{x}^*$ corresponding to d^*. We finally note that there cannot be an equilibrium point on the boundary of the set Ω since the system trajectory cannot stay on the boundary indefinitely as we will show in Section III-B.

Proposition III.1. *Let $0 \le \alpha_{i,min} \le \alpha_i \le \alpha_{i,max}, \ \forall i \in \mathcal{M}$, where the elements of the vector α_{max} are arbitrarily large. If α_{min} and d_{max} satisfy the condition (11), then there exists a unique equilibrium solution, (x^*, d^*), to the system (7) on the set Ω, which is further an inner point of Ω.*

Remark III.2. In the special case of logarithmic utility functions, $U_i(x_i) = u_i \log(x_i + 1)$, condition (11) can be explicitly given as

$$0 < \frac{1}{C+M} \sum_{i=1}^M \frac{u_i}{\alpha_i} < d_{max}.$$

In addition, as $d_{max} \to \infty$ the system admits a unique inner equilibrium solution for any finite value of α.

We now define the system around the unique inner equilibrium point:

$$\begin{aligned} \dot{\tilde{x}}_i(t) &= g_i(\tilde{x}_i) - \alpha_i \tilde{d}(t) \ , \ i = 1, \ldots, M, \\ \dot{\tilde{d}}(t) &= \frac{1}{C} \sum_{i=1}^M \tilde{x}_i, \end{aligned} \quad (12)$$

where the functions $g_i(\tilde{x}_i)$ are defined as

$$g_i(\tilde{x}_i) := \frac{dU_i(x_i)}{dx_i} - \frac{dU_i(x_i^*)}{dx_i}.$$

Note that

$$g_i(\tilde{x}_i) \begin{cases} > 0 & , \text{ if } \tilde{x}_i < 0 \\ < 0 & , \text{ if } \tilde{x}_i > 0 \\ = 0 & , \text{ if } \tilde{x}_i = 0, \end{cases} \tag{13}$$

due to the fact that $U_i(x_i)$ is strictly concave in x_i, and hence, $(dU_i(x_i)/dx_i)$ is strictly decreasing.

Define next an energy-like Lyapunov function:

$$V(\tilde{\mathbf{x}}, \tilde{d}) = \sum_{i=1}^{M} \frac{1}{\alpha_i} (\tilde{x}_i)^2 + C(\tilde{d})^2 , \tag{14}$$

which is positive definite.

The derivative of V along the system trajectories is given by $\dot{V} = \sum_{i=1}^{M} (2/\alpha_i) g_i(\tilde{x}_i)\tilde{x}_i \leq 0$, and is equal to zero only if $\tilde{x}_i = 0 \, \forall i \Rightarrow \tilde{d} = 0$. Let $S := \{(\tilde{\mathbf{x}}, \tilde{d}) \in \mathbb{R}^{M+1} : \dot{V}(\tilde{\mathbf{x}}, \tilde{d}) = 0\}$. It follows from (13) that $S = \{(\tilde{\mathbf{x}}, \tilde{d}) \in \mathbb{R}^{M+1} : \tilde{\mathbf{x}} = 0\}$. Hence, for any trajectory of the system that belongs to S, we have $\tilde{\mathbf{x}} \equiv 0$. It follows then directly from (12) and the fact that $g_i(0) = 0 \ \forall i$ that $\tilde{\mathbf{x}} \equiv 0 \Rightarrow \dot{\tilde{x}}_i = 0 \ \forall i \Rightarrow \tilde{d} = 0$. Therefore, the only solution that can stay identically in S is the zero solution, which corresponds to the unique equilibrium of the original system (7). Furthermore, it is globally asymptotically stable by LaSalle's invariance theorem [31]. This completes the proof of stability without the boundary effects. Since global stability under the boundary effects for a more general case will be provided in Theorem III.7. We postpone the treatment of boundaries until then.

B. Stability for a General Network Topology with Multiple Users

We now establish the stability of the system under a general network topology with multiple links, and with a general routing matrix $\mathbf{A}$ as defined in (1). The generalized system is described by

$$\begin{aligned} \dot{x}_i(t) &= \frac{dU_i(x_i)}{dx_i} - \alpha_i D_i(t) \ , \ i = 1, \ldots, M \ , \\ \dot{d}_l(t) &= \frac{\bar{x}_l}{C_l} - 1 \ , \ l = 1, \ldots, L \ , \end{aligned} \tag{15}$$

with the boundary behavior given by (4) and (6). Define the feasible set Ω (as before) as

$$\begin{aligned} \Omega = \{(\mathbf{x}, \mathbf{d}) \in \mathbb{R}^{M+L} : 0 \leq x_i \leq x_{i,max} \\ \text{and } 0 \leq d_l \leq d_{l,max} \ , \ \forall i \ , l\}, \end{aligned}$$

where $d_{l,max}$ and $x_{i,max}$ are upper bounds on d_l and x_i, respectively. Define $\mathbf{d}_{max} := [d_{1,max}, \ldots, d_{L,max}]$. We first investigate existence and uniqueness of an inner equilibrium on the set Ω under the assumption of $x_{i,max} > C_l, \forall l$. Toward this end, we assume that $\mathbf{A}$ is a full row rank matrix with $M \geq L$, which is in fact no loss of generality as non-bottleneck links on the network have no effect on the equilibrium point, and can safely be left out.

The study of existence follows lines similar to the one in the case of a single link. Supposing that (15) admits an inner equilibrium and by setting $\dot{x}_i(t)$ and $\dot{d}_l(t)$ equal to zero for all l and i one obtains

$$\mathbf{A}\,\mathbf{x} = \mathbf{C} \tag{16}$$

$$\mathbf{f}(\alpha, \mathbf{x}) = \mathbf{A}^T \mathbf{d} \ \ , \tag{17}$$

where $\mathbf{d} := [d_1, \ldots, d_L]^T$ is the delay vector at the links, $\mathbf{C}$ is the capacity vector introduced earlier, and the nonlinear vector function $\mathbf{f}$ is defined as

$$\mathbf{f}(\alpha, \mathbf{x}) := \left[\frac{1}{\alpha_1} \frac{dU_1}{dx_1}, \ldots, \frac{1}{\alpha_M} \frac{dU_M}{dx_M} \right]^T . \tag{18}$$

Define $X := \{\mathbf{x} \in \mathbb{R}^M : \mathbf{A}\mathbf{x} = \mathbf{C}\}$ as the set of flows, $\mathbf{x}$, which satisfy (16).

Multiplying (17) from left by $\mathbf{A}$ yields

$$\mathbf{A}\,\mathbf{f}(\alpha, \mathbf{x}^*) = \mathbf{A}\mathbf{A}^T \mathbf{d}.$$

Since $\mathbf{A}$ is of full row rank, the square matrix $\mathbf{A}\mathbf{A}^T$ is full rank, and hence invertible. Thus, for a given flow vector $\mathbf{x}$ and pricing vector α,

$$\mathbf{d}(\alpha, \mathbf{x}) = (\mathbf{A}\mathbf{A}^T)^{-1} \mathbf{A} \mathbf{f}(\alpha, \mathbf{x}), \tag{19}$$

is unique. From the definition of $\mathbf{f}$, $\mathbf{d}(\alpha, \mathbf{x})$ is a linear combination of $p_i(x_i)/\alpha_i$, and hence, strictly decreasing in α. Since the set X is compact, the continuous function $\mathbf{d}(\alpha, \mathbf{x})$ admits a maximum value on the set X for a given α. Therefore, for each $\epsilon > 0$ one can choose the elements of α_{max} sufficiently large such that

$$0 < \max_{\mathbf{x} \in X} \mathbf{d}(\alpha_{max}, \mathbf{x}) < \epsilon.$$

In addition, given X and $\mathbf{d}_{max}$, one can find α_{min} such that

$$0 < \max_{\mathbf{x} \in X} \mathbf{d}(\alpha_{min}, \mathbf{x}) < \mathbf{d}_{max}, \tag{20}$$

Hence, we conclude that there is at least one inner equilibrium solution, $(\mathbf{x}^*, \mathbf{d}^*)$, on the set Ω, which satisfies (16) and (17).

We next establish the uniqueness of the equilibrium. Suppose that there are two different equilibrium points, $(\mathbf{x}_1^*, \mathbf{d}_1^*)$ and $(\mathbf{x}_2^*, \mathbf{d}_2^*)$. Then, from (16) it follows that

$$\mathbf{A}\,(\mathbf{x}_1^* - \mathbf{x}_2^*) = 0 \ \Leftrightarrow \ (\mathbf{x}_1^* - \mathbf{x}_2^*)^T \mathbf{A}^T = 0$$

Similarly, from (17) we have

$$\mathbf{f}(\alpha, \mathbf{x}_1^*) - \mathbf{f}(\alpha, \mathbf{x}_2^*) = \mathbf{A}^T (\mathbf{d}_1^* - \mathbf{d}_2^*) \ .$$

Multiplying this with $(\mathbf{x}_1^* - \mathbf{x}_2^*)^T$ from left we obtain

$$(\mathbf{x}_1^* - \mathbf{x}_2^*)^T \left[\mathbf{f}(\alpha, \mathbf{x}_1^*) - \mathbf{f}(\alpha, \mathbf{x}_2^*)\right] = 0$$

We rewrite this as

$$\sum_{i=1}^{M} (\mathbf{x}_{1i}^* - \mathbf{x}_{2i}^*) \frac{1}{\alpha_i} \left[\frac{dU_i(x_{1i}^*)}{dx_i} - \frac{dU_i(x_{2i}^*)}{dx_i} \right] = 0.$$

Since U_i's are strictly concave, each term (say the i-th one) in the summation is negative whenever $x_{1\,i}^* \neq x_{2\,i}^*$ with equality holding only if $x_{1i}^* = x_{2i}^*$. Hence, we conclude that x^* has to be unique, that is $\mathbf{x}^* = \mathbf{x}_1^* = \mathbf{x}_2^*$. From this, and (15), it immediately follows that $D_i, i = 1, \ldots, M$, are unique. This does not however immediately imply that $d_l, l = 1, \ldots, L$, are also unique, which in fact may not be the case if $\mathbf{A}$ is not full row rank. The uniqueness of d_l's, however, follow from (19), where we obtain a unique $\mathbf{d}^*$ for a given equilibrium flow vector $\mathbf{x}^*$:

$$\mathbf{d}^* = (\mathbf{A}\mathbf{A}^T)^{-1}\mathbf{A}\mathbf{f}(\alpha, \mathbf{x}^*).$$

As a result, $(\mathbf{x}^*, \mathbf{d}^*)$, obtained from (16) and (17) constitutes a unique inner equilibrium point on the set Ω.

Proposition III.3. *Let* $0 \le \alpha_{i,min} \le \alpha_i \le \alpha_{i,max}$, $\forall i \in \mathcal{M}$ *where the elements of the vector* α_{max} *are arbitrarily large, and* $\mathbf{A}$ *be of full row rank. Given* X*, if* α_{min} *and* $\mathbf{d}_{max}$ *satisfy*

$$0 < \max_{\mathbf{x} \in X} \mathbf{d}(\alpha_{min}, \mathbf{x}) < \mathbf{d}_{max},$$

where $\mathbf{d}(\alpha, \mathbf{x})$ *is defined in (19), then the system (15) has a unique equilibrium point,* $(\mathbf{x}^*, \mathbf{d}^*)$*, which is in the interior of the set* Ω.

Defining the delays at links, d_l, and user flow rates, x_i, around the equilibrium as $\tilde{d}_l := d_l - d_l^*$ and $\tilde{x}_i := x_i - x_i^*$, respectively, for all l and i, we obtain the following system inside the set Ω and around the equilibrium:

$$\begin{aligned} \dot{\tilde{x}}_i(t) &= g_i(\tilde{x}_i) - \alpha_i \tilde{D}_i(t) \ , \ i = 1, \ldots, M \ , \\ \dot{\tilde{d}}_l(t) &= \frac{1}{C_l} \sum_{i: l \in R_i} \tilde{x}_i \ , \ l = 1, \ldots, L \ , \end{aligned} \tag{21}$$

where $\tilde{D}_i = \sum_{l \in R_i} \tilde{d}_l$, and $g_i(.)$ is defined as in (13).

We define a positive definite Lyapunov function as a generalized version of (14):

$$V(\tilde{\mathbf{x}}, \tilde{\mathbf{d}}) = \sum_{i=1}^{M} \frac{1}{\alpha_i} (\tilde{x}_i)^2 + \sum_{l=1}^{L} C_l (\tilde{d}_l)^2 \tag{22}$$

The time derivative of $V(\tilde{\mathbf{x}}, \tilde{\mathbf{d}})$ along the system trajectories is given by $\dot{V}(\tilde{\mathbf{x}}, \tilde{\mathbf{d}}) = \sum_{i=1}^{M} (2/\alpha_i) g_i(\tilde{x}_i)\, \tilde{x}_i \le 0$, where the inequality follows because $g_i(\tilde{x}_i)\, \tilde{x}_i \le 0 \, \forall i$. Thus, $\dot{V}(\tilde{\mathbf{x}}, \tilde{\mathbf{d}})$ is negative semidefinite. Let $S := \{(\tilde{\mathbf{x}}, \tilde{\mathbf{d}}) \in \mathbb{R}^{M+L} : \dot{V}(\tilde{\mathbf{x}}, \tilde{\mathbf{d}}) = 0\}$. It follows as before that $S = \{(\tilde{\mathbf{x}}, \tilde{\mathbf{d}}) \in \mathbb{R}^{M+L} : \tilde{\mathbf{x}} = 0\}$. Hence, for any trajectory of the system that belongs identically to the set S, we have $\tilde{\mathbf{x}} = 0$. It follows directly from (21) and the fact that $g_i(0) = 0 \, \forall i$ that

$$\tilde{\mathbf{x}} = 0 \Rightarrow \dot{\tilde{\mathbf{x}}} = 0 \Rightarrow \tilde{D}_i = 0 \, \forall i \Rightarrow \tilde{d}_l = 0 \, \forall l,$$

where the last implication is due to the fact that $\tilde{D} = \mathbf{A}^T \, \tilde{\mathbf{d}}^*$ and the matrix $\mathbf{A}$ is of full row rank. Therefore, the only solution that can stay identically in S is the zero solution, which corresponds to the unique inner equilibrium of the original system.

Let us redefine the feasible set in terms of the tilde'd quantities, $\tilde{\mathbf{x}}$ and $\tilde{\mathbf{d}}$, as follows:

$$\begin{aligned} \tilde{\Omega} := \{(\tilde{\mathbf{x}}, \tilde{\mathbf{d}}) \in \mathbb{R}^{M+L} &: -x_i^* \le \tilde{x}_i \le x_{i,max} - x_i^* \\ &\text{and } -d_l^* \le \tilde{d}_l \le d_{l,max} - d_l^* \ , \ \forall i \ , l\}. \end{aligned}$$

We note that the set $\tilde{\Omega}$ can also be defined through a set of (linear) inequalities $h_j(\tilde{\mathbf{x}}, \tilde{\mathbf{d}}) \le 0, \ j \in \mathcal{H} := \{1, \ldots, H\}$. Given X, let $\mathbf{d}_{max}$ and α_{min} be chosen such that (20) holds. Then, by Proposition III.3 the unique equilibrium point, $(0, 0)$, is an inner solution on $\tilde{\Omega}$, where $h_j(0, 0) < 0 \, \forall j \in \mathcal{H}$.

We now investigate the effect of the boundaries given in $\tilde{\Omega}$ and described by (4) and (6). The system (21) can also be written compactly as

$$\dot{\mathbf{z}} := \begin{pmatrix} \dot{\tilde{\mathbf{x}}} \\ \dot{\tilde{\mathbf{d}}} \end{pmatrix} = \mathbf{F} \begin{pmatrix} \tilde{\mathbf{x}} \\ \tilde{\mathbf{d}} \end{pmatrix} := \mathbf{F}(\mathbf{z}), \tag{23}$$

with the definition of $\mathbf{F}$ being obvious from (21). We have shown earlier that $\dot{V} \le 0$ when the system trajectory is strictly inside the set $\tilde{\Omega}$. On the other hand, if the trajectory hits the boundary, then it stays on the boundary as long as the system gradient $\mathbf{F}(\mathbf{z})$ in (23) points toward the boundary. This is known as *sliding mode* behavior, which we define in this context as follows:

Definition III.4. Let the gradient vector and the corresponding unit vector of the boundary surface $\{\mathbf{z} : h(\mathbf{z}) = 0\}$ be given by $\mathbf{n}(\mathbf{z}) := \nabla h(\mathbf{z})$ and $\mathbf{n}_u(\mathbf{z}) := \mathbf{n}(\mathbf{z})/||\mathbf{n}(\mathbf{z})||$, respectively. Suppose that the trajectory of the system $\dot{\mathbf{z}} = \mathbf{F}(\mathbf{z})$ is *inside* the surface, $h(\mathbf{z}) \le 0 \ \forall \mathbf{z}$. Then, the system exhibits *sliding mode* behavior on the boundary surface $\{\mathbf{z} : h(\mathbf{z}) = 0\}$, if and only if,

$$\mathbf{n}(\mathbf{z}) \cdot \mathbf{F}(\mathbf{z}) \ge 0,$$

where $\mathbf{n} \cdot \mathbf{F}$ denotes the dot product of the vectors $\mathbf{n}$ and $\mathbf{F}$. Furthermore, the gradient of the system trajectory during sliding mode is given by

$$\mathbf{F}_s := \mathbf{F} - (\mathbf{F} \cdot \mathbf{n}_u)\mathbf{n}_u \, .$$

We next establish the relationship between the Lyapunov function $V(\mathbf{z})$ in (22) and the sliding mode behavior of the trajectory along the boundaries.

Proposition III.5. *Assume that the system (23) has a unique inner equilibrium point,* $\mathbf{z}^*$*, in the compact and convex feasible set* $\tilde{\Omega}$*, and*

$$\dot{V}(\mathbf{z})|_{trajectory} := \nabla_{\mathbf{z}} V(\mathbf{z}) \cdot \mathbf{F}(\mathbf{z}) \le 0,$$

for all $\mathbf{z}$ *such that* $h_j(\mathbf{z}) < 0 \forall j \in \mathcal{H}$*, where the Lyapunov function* $V(\mathbf{z})$ *is given by (22). The system trajectory exhibits sliding mode behavior along the j-*th *boundary* $\{\mathbf{z} : h_j(\mathbf{z}) = 0\}$ $j \in \mathcal{H}$*, if and only if,*

$$\dot{V}(\mathbf{z})|_{sliding} \le \dot{V}(\mathbf{z})|_{trajectory} \le 0, \tag{24}$$

where $\dot{V}(\mathbf{z})|_{sliding} := \nabla_{\mathbf{z}} V(\mathbf{z}) \cdot \mathbf{F}_s(\mathbf{z})$, $\forall \mathbf{z} \in \{\mathbf{z} : h_j(\mathbf{z}) = 0\}$.

Proof. It follows from the definition of $\mathbf{F}_s$ that

$$\dot{V}|_{sliding} = \nabla_{\mathbf{z}} V(\mathbf{z}) \cdot [\mathbf{F} - (\mathbf{F} \cdot \mathbf{n}_u)\mathbf{n}_u] \, , \tag{25}$$

where we drop the arguments $\mathbf{z}$ and j to simplify the notation. From (22), $\nabla_{\mathbf{z}} V(\mathbf{z})$ is given by

$$\nabla_{\mathbf{z}} V(\mathbf{z}) = \left[\frac{2\tilde{x}_1}{\alpha_1}, \ldots, \frac{2\tilde{x}_M}{\alpha_M}, 2C_1 d_1, \ldots, 2C_L d_L \right].$$

On the other hand, by the definition of $\tilde{\Omega}$, the vector $\mathbf{n}$, of length $M + L$, has all zero entries except the k-th one, which is

$$n_k = \begin{cases} \text{sign}(\tilde{\mathrm{x}}_{\mathrm{k}}), & \text{if } 1 \le k \le M \\ \text{sign}(\tilde{\mathrm{d}}_{\mathrm{k}}), & \text{if } M + 1 \le k \le M + L, \end{cases}$$

where $sign(\cdot)$ is the sign function. Note that, $\mathbf{n}_u = \mathbf{n}$ by definition. Thus, on the boundary $\{\mathbf{z} : h(\mathbf{z}) = 0\}$ the term $\nabla_{\mathbf{z}} V(\mathbf{z}) \cdot \mathbf{n}_u$ yields either $|\tilde{x}_k|$ or $\left|\tilde{d}_k\right|$, and we obtain

$$\nabla_{\mathbf{z}} V(\mathbf{z}) \cdot \mathbf{n}_u \geq 0.$$

We now show the necessity: if the trajectory exhibits sliding mode behavior along the boundary, then we have $\mathbf{n}(\mathbf{z}) \cdot \mathbf{F}(\mathbf{z}) \geq 0$ and $\nabla_{\mathbf{z}} V(\mathbf{z}) \cdot \mathbf{n}_u \geq 0$. Hence, the result in (24) follows immediately from (25). Next, in order to show the sufficiency we note that (24) yields

$$\nabla_{\mathbf{z}} V(\mathbf{z}) \cdot (\mathbf{F} \cdot \mathbf{n}_u)\mathbf{n}_u \geq 0.$$

Since $\nabla_{\mathbf{z}} V(\mathbf{z}) \cdot \mathbf{n}_u \geq 0$ is already established on the boundary, we immediately obtain $\mathbf{n} \cdot \mathbf{F} \geq 0$, which corresponds to sliding mode behavior of the trajectory. ☐

Now, as a corollary to this proposition, we have the following result.

Corollary III.6. *Consider the system (23) on the feasible set $\tilde{\Omega}$ with the unique inner equilibrium point $\mathbf{z}^*$. Furthermore, let the time derivative of the Lyapunov function V of (22) be non-increasing along the system trajectory without boundary effect. Then, the system trajectory exhibits sliding mode behavior, if and only if, the rate of decrease in V at the boundary point is less than or equal to the one without the boundary effect.*

From Proposition III.5, we have $\dot{V} \leq 0$ for all $(\tilde{\mathbf{x}}, \tilde{\mathbf{d}}) \in \tilde{\Omega}$. It was already shown that the unique inner equilibrium solution, $(\mathbf{x}^*, \mathbf{d}^*)$, is the only invariant solution in $S \cap \tilde{\Omega}$. Thus, the asymptotic stability of the system follows from LaSalle's invariance theorem. This is captured in the following theorem.

Theorem III.7. *Let $0 \leq \alpha_{min} \leq \alpha \leq \alpha_{max}$, where α_{max} is arbitrarily large, and $\mathbf{A}$ be of full row rank. Given X, suppose that α_{min} and $\mathbf{d}_{max}$ are chosen such that*

$$0 < \max_{\mathbf{x} \in X} \mathbf{d}(\alpha_{min}, \mathbf{x}) < \mathbf{d}_{max},$$

holds, and hence, the system

$$\dot{x}_i(t) = \frac{dU_i(x_i)}{dx_i} - \alpha_i D_i(t)\ ,\ i = 1, \ldots, M\ ,$$
$$\dot{d}_l(t) = \frac{\bar{x}_l}{C_l} - 1\ ,\ l = 1, \ldots, L\ ,$$

admits the unique inner equilibrium point $(\mathbf{x}^*, \mathbf{d}^*)$. Then, this system with its boundary point behavior described by (4) and (6) is globally asymptotically stable on the set

$$\Omega := \{(x, \mathbf{d}) \in \mathbb{R}^{M+L} : 0 \leq x_i \leq x_{i,max} \text{ and } 0 \leq d_l \leq d_{l,max},\ \forall i\ , l\}. \tag{26}$$

We note that the unique equilibrium point of the system is only an approximation to the Nash equilibrium since the effect of the i^{th} user on the delay, $D_i(\mathbf{x}, t)$, s/he experiences has been ignored. This approximation becomes more accurate as the number of users in the network increases.

Remark III.8. The unique equilibrium solution (x^*, d^*) of the system (15) solves the following resource allocation problem

$$\max_{(\mathbf{x},\mathbf{d})} \sum_{i=1}^{M} \frac{1}{\alpha_i} U_i(x_i) - \sum_{l=1}^{L} \int_0^{\bar{x}_l} d_l(\tau) d\tau,$$

which is a relaxed and scaled version of the original optimization problem $\max_{\mathbf{x}} \sum_{i=1}^{M} U_i(x_i)$ of [2] under link capacity and positivity of flow rate conditions. This is due to the fact that the partial derivatives of the costs at the links, and not on the path of the i-th user, with respect to x_i yield zero. Likewise, the utility function of each user depends only on that user's flow rate. In addition, we have $\mathbf{Ax} = \mathbf{C}$ at the equilibrium point, $(\mathbf{x}^*, \mathbf{d}^*)$, i.e. full capacity usage. Furthermore, for $U_i = w_i \log(x_i)$, where w_i is a user specific preference parameter, the unique system equilibrium $(\mathbf{x}^*, \mathbf{d}^*)$ approximates a proportionally fair allocation [3], [4]. ☐

IV. Stability under Information Delay

We have shown in Section III that the system described by (4) and (6) is globally asymptotically stable under a general network topology. We now investigate the global stability of the system under arbitrary propagation delays, which we are also going to refer to as *information delay*, and denote by r. First, we analyze the simple case of a single link with a single user to gain insight into the problem. Next, we generalize the analysis to a general network with a single bottleneck node and multiple users. We do not consider the case of multiple users on a general network topology with multiple links, since the problem in that case is quite intractable under arbitrary information delays.

A. Stability for a Single Link with a Single User under Information Delay

For the case of a single user on a single link, we describe the system by introducing a maximum propagation (information) delay r between the user and the link, which we assume to be a constant :

$$\begin{aligned} \dot{x}(t) &= \frac{dU(x(t))}{dx} - \alpha d(t - r) \\ \dot{d}(t) &= \frac{x(t - r)}{C} - 1 \end{aligned} \tag{27}$$

We assume that $\alpha > \frac{1}{d_{max}} \frac{dU(C)}{dx}$, and $x_{max} > C$, so that $x^* = C$, $d^* = \frac{1}{\alpha} \frac{dU(C)}{dx}$ is the unique equilibrium of (27), which is in the interior of the feasible set.

Defining the queueing delay and flow rate around the unique inner equilibrium point as in the no delay case, we obtain the following equivalent system:

$$\begin{aligned} \dot{\tilde{x}}(t) &= g(\tilde{x}(t)) - \alpha \tilde{d}(t - r) \\ \dot{\tilde{d}}(t) &= \frac{1}{C} \tilde{x}(t - r) \end{aligned} \tag{28}$$

Notice that (28) is a set of delay differential equations. Such systems have been studied extensively in the literature; see, e.g. [32], [33]. Here we will make particular use of the methods presented in Chapter 4.2 of [33]. From (28), we immediately have

$$\dot{\tilde{x}}(t) = g(\tilde{x}(t)) - \alpha \tilde{d}(t + r) + \alpha[\tilde{d}(t + r) - \tilde{d}(t - r)],$$

and

$$\dot{\tilde{x}}(t - r) = g(\tilde{x}(t - r)) - \alpha \tilde{d}(t) + \frac{\alpha}{C} \int_{-2r}^{0} \tilde{x}(t + s) ds\,.$$

We define a Lyapunov function

$$V(\tilde{x},\tilde{d}) = \frac{1}{\alpha}(\tilde{x}(t-r))^2 + C(\tilde{d}(t))^2 + \frac{1}{C}\int_{-2r}^{0}\int_{t+s}^{t}\tilde{x}^2(u-r)du\, ds, \tag{29}$$

which is positive definite. Taking the time-derivative of V along the system trajectories, we obtain

$$\begin{aligned}\dot{V}(\tilde{x},\tilde{d}) = &\frac{2}{\alpha}g(\tilde{x}(t-r))\tilde{x}(t-r)\\ &+\frac{2}{C}\int_{-2r}^{0}\tilde{x}(t-r)\tilde{x}(t+s-r)ds\\ &+\frac{1}{C}\int_{-2r}^{0}[\tilde{x}^2(t-r)-\tilde{x}^2(t+s-r)]ds\end{aligned}$$

Using the simple algebraic inequality

$$2\tilde{x}(t-r)\tilde{x}(t+s-r) \le \tilde{x}^2(t-r)+\tilde{x}^2(t+s-r),$$

one can bound $\dot{V}$ above by

$$\dot{V}(\tilde{x},\tilde{d}) \le \frac{2}{\alpha}g(\tilde{x}(t-r))\tilde{x}(t-r) + \frac{4r}{C}\tilde{x}^2(t-r)$$

Thus, $\dot{V}(\tilde{x},\tilde{d})$ can be made negative semi-definite by imposing a condition on the maximum delay r. In this case, let $S := \{(\tilde{x},\tilde{d}) \in \tilde{\Omega} : \dot{V}(\tilde{x},\tilde{d}) = 0\}$. It follows immediately that $S = \{(\tilde{x},\tilde{d}) \in \tilde{\Omega} : \tilde{x} = 0\}$. Hence, for any trajectory of the system that belongs to S, we have $\tilde{x} \equiv 0$. It also follows directly from (28), since $g(0) = 0$, that

$$\tilde{x} \equiv 0 \Rightarrow \dot{\tilde{x}} = 0 \Rightarrow \tilde{d} = 0\ .$$

Therefore, the only solution that can stay identically in S is the zero solution, which corresponds to the unique equilibrium of the original system.

We thus conclude that the system (28) is asymptotically stable by LaSalle's invariance theorem if the maximum delay r satisfies the condition

$$r < \frac{C}{2\alpha}k\,, \tag{30}$$

where k is defined as

$$k := \inf_{-C\le\tilde{x}\le x_{max}-C}\left|\frac{g(\tilde{x})}{\tilde{x}}\right|.$$

In order to gain further insight into this condition, we compute the parameter k for the specific case when the utility function is taken as the logarithmic one, that is $U(x) = u\log(x+1)$. In this case we obtain

$$g(\tilde{x}) = \frac{u}{x+1} - \frac{u}{C+1} = \frac{-u\tilde{x}}{(x+1)(C+1)}$$

and hence

$$k = \min_{0\le x\le x_{max}}\frac{u}{(x+1)(C+1)} = \frac{u}{(x_{max}+1)(C+1)}.$$

Hence, a safe bound on r is

$$r < \frac{uC}{2\alpha(x_{max}+1)(C+1)}.$$

Of course a better bound can be obtained if we know that the trajectory remains in a small neighborhood of the equilibrium, C. This would very much be dependent on the application at hand.

The analysis of the effect of boundaries on system stability is almost identical to the one of the case without delay, and we again make use of Proposition III.5. This now brings us to the following theorem, where again LaSalle's invariance theorem is invoked:

Theorem IV.1. *Let $\alpha > \frac{1}{d_{max}}\frac{dU(C)}{dx}$, and $x_{max} > C$. Then, the system*

$$\begin{aligned}\dot{x}(t) &= \frac{dU(x(t))}{dx} - \alpha d(t-r)\\ \dot{d}(t) &= \frac{1}{C}x(t-r) - 1,\end{aligned}$$

with the unique inner equilibrium point ($x^* = C, d^* = \frac{1}{\alpha}\frac{dU(C)}{dx}$) and boundary point behavior described by the delayed versions of (4) and (6) is globally asymptotically stable on the set Ω if the maximum delay, r, in the system satisfies the condition

$$r < \frac{kC}{2\alpha}\,,$$

where $k := \inf_{-x^*\le\tilde{x}\le x_{max}-x^*}|g(\tilde{x})/\tilde{x}|$, and $x^* = C$.

B. Stability for a Single (Bottleneck) Link with Multiple Users under Information Delay

We now generalize the preceding analysis of a single link with a single user to multiple users by introducing user specific maximum propagation delays $r = [r_1, \ldots, r_M]$ between the link and the users. We invoke the assumptions of Section III so that the system has a unique inner equilibrium point $(\mathbf{x}^*, d^*)$ as characterized in Section III. Modifying the system equations (12) around this equilibrium point by introducing the associated maximum propagation delays, we obtain

$$\begin{aligned}\dot{\tilde{x}}_i(t) &= g_i(\tilde{x}_i(t)) - \alpha_i\tilde{d}(t-r_i)\ ,\ i = 1,\ldots,M\\ \dot{\tilde{d}}(t) &= \frac{1}{C}\sum_{i=1}^{M}\tilde{x}_i(t-r_i)\end{aligned} \tag{31}$$

Following an approach similar to the one in the single user case with delay, one gets for the i^{th} user

$$\begin{aligned}\dot{\tilde{x}}_i(t-r_i) = &\ g_i(\tilde{x}_i(t-r_i)) - \alpha_i\tilde{d}(t)\\ &+\frac{\alpha_i}{C}\int_{-2r_i}^{0}\sum_{j=1}^{M}\tilde{x}_j(t+s-r_j)ds.\end{aligned}$$

We again define a positive definite Lyapunov function:

$$\begin{aligned}V(\tilde{\mathbf{x}},\tilde{d}) = &\textstyle\sum_{i=1}^{M}\frac{1}{\alpha_i}(\tilde{x}_i(t-r_i))^2 + C(\tilde{d}(t))^2\\ &+\frac{M}{C}\textstyle\sum_{i=1}^{M}\int_{-2r_i}^{0}\int_{t+s}^{t}\tilde{x}_i^2(u-r_i)du\, ds.\end{aligned} \tag{32}$$

Taking the derivative of V along the system trajectories, we obtain

$$\begin{aligned}\dot{V}(\tilde{\mathbf{x}},\tilde{d}) = &\textstyle\sum_{i=1}^{M}\frac{2}{\alpha_i}g_i(\tilde{x}_i(t-r_i))\tilde{x}(t-r_i)\\ &+\frac{1}{C}\textstyle\int_{-2r_i}^{0}\sum_{i=1}^{M}\sum_{j=1}^{M}2\tilde{x}_i(t-r_i)\tilde{x}_j(t+s-r_j)ds\\ &+\frac{M}{C}\textstyle\sum_{i=1}^{M}\int_{-2r_i}^{0}[\tilde{x}_i^2(t-r)-\tilde{x}_i^2(t+s-r)]ds\end{aligned}$$

We bound the derivative $\dot{V}$ from above by

$$\dot{V}(\tilde{\mathbf{x}}, \tilde{d}) \leq \sum_{i=1}^{M} \frac{2}{\alpha_i} g_i(\tilde{x}_i(t-r_i))\tilde{x}_i(t-r_i) + \frac{4Mr_i}{C}\tilde{x}_i^2(t-r_i)$$

This can be made negative semi-definite by imposing a condition on the maximum delay in the system, $r_{max} := \max_i r_i$. Let $S := \{(\tilde{\mathbf{x}}, \tilde{d}) \in \tilde{\Omega} : \dot{V}(\tilde{\mathbf{x}}, \tilde{d}) = 0\}$. It follows as before that $S = \{(\tilde{\mathbf{x}}, \tilde{d}) \in \tilde{\Omega} : \tilde{\mathbf{x}} = 0\}$. Hence, for any trajectory of the system that belongs identically to the set S, we have $\tilde{\mathbf{x}} = 0$. It also follows directly from (31), and the fact that $g_i(0) = 0 \; \forall i$, that

$$\tilde{\mathbf{x}} = 0 \Rightarrow \dot{\tilde{\mathbf{x}}} = 0 \Rightarrow \tilde{d} = 0,$$

where we have made use of the fact that the matrix $\mathbf{A}$ is of full row rank. Therefore, the only solution that can stay identically in S is the zero solution, which corresponds to the unique equilibrium of the original system. As a result, the system (31) is asymptotically stable by LaSalle's invariance theorem if the maximum delay in the system, r_{max}, satisfies the condition

$$r_{max} < \frac{k_{min}}{2\alpha_{max}} \frac{C}{M}, \qquad (33)$$

where α_{max} and k_{min} are defined as

$$\alpha_{max} := \max_i \alpha_i$$
$$k_{min} := \min_i \inf_{-x_i^* \leq \tilde{x}_i \leq x_{i,max} - x_i^*} \left| \frac{g(\tilde{x}_i)}{\tilde{x}_i} \right| \qquad (34)$$

Notice that the bound on the maximum delay required for the stability of the system is affected by, among other things, the maximum pricing parameter and the capacity per user C/M. Since the link capacity C will be provisioned in the network design stage according to the expected maximum number of users the proposed algorithm is in practice scalable for the given capacity per user.

The analysis of the boundary effects is identical to earlier ones, and therefore will be omitted. The following theorem now extends the results of Theorem IV.1 to the multi-user case.

Theorem IV.2. *Let the conditions in Theorem III.7 hold such that the system*

$$\dot{x}_i(t) = \frac{dU_i(x_i(t))}{dx_i} - \alpha_i d(\mathbf{x}, t-r), \; i = 1, \ldots, M,$$
$$\dot{d}(t) = \frac{1}{C}\sum_{i=1}^{M} x_i(t-r_i) - 1,$$

with the boundary point behavior described by (4) and (6) admits a unique inner equilibrium point $(\mathbf{x}^*, d^*)$. *This system is globally asymptotically stable on the corresponding set* Ω *defined by (26), if the maximum delay,* r_{max}*, in the system satisfies the condition*

$$r_{max} < \frac{k_{min}}{2\alpha_{max}} \frac{C}{M},$$

where α_{max} *and* k_{min} *are defined in (34).*

V. An Adaptive Pricing Scheme

In Sections III and IV, we have shown that the pricing parameter α can be chosen such that the equilibrium point is feasible, $\mathbf{d}^* < \mathbf{d}_{max}$, if $\mathbf{x}^* < \mathbf{x}_{max}$. In other words, we have assumed either the equilibrium queue sizes (delays) are less than the maximum queue size (delay) or there are infinite buffers at all nodes. In fact, the buffer sizes on a real network are limited, and choosing the parameter α appropriately in a dynamic networking environment with varying numbers of users and changing preferences is not a trivial task. If α is chosen to be "too high", then the proposed congestion control scheme becomes less robust. On the other hand, a "too small" α may lead to large queueing delays as well as packet losses. Furthermore, it is not possible to choose α in a centralized manner due to communication constraints.

Given the capacity vector $\mathbf{C}$, it is possible to derive $\mathbf{d}_{max}$ directly from maximum queue sizes at the network nodes. Consequently, the vector $\mathbf{d}_{max}$ denotes a strict physical bound on the maximum queueing delays. The nonlinear vector function $\mathbf{f}$ is strictly decreasing in α by (18). Hence, from (19) the unique equilibrium $\mathbf{d}^*$ is strictly decreasing in α. Then, a natural strategy for making the equilibrium queue size at node l, d_l^*, feasible is to increase α_i for all users whose flows pass through link l. At the same time, those users will experience packet losses at node l if $d_l^* > d_{l,max}$, i.e. equilibrium queue size (delay) is larger than the maximum one. By making use of the packet losses experienced by the users as a signal to increase prices it is possible to minimize packet losses. On the other hand, α should not be "too high" for the congestion control scheme to be robust. Thus, we slowly decrease α until a network-wide lower bound is reached unless there is a packet loss. As a result, we obtain a distributed, dynamic, and adaptive pricing scheme, which improves robustness of the congestion control scheme to variations in network parameters, such as the number of users, user preferences, link capacities, etc. We also note that if a virtual queueing scheme [9] is implemented at the buffers and marked packets are used as the signal, then one can prevent packet losses altogether, and set an upper-bound on the maximum delay in the system.

A. Hybrid Modeling of the Adaptive Pricing Scheme

We model the proposed adaptive pricing scheme as a *switched (hybrid) system*, and analyze its stability. In this context, a switched hybrid system can be defined as a continuous-time system with isolated discrete switching events [34]. Let us first consider for illustrative purposes the case of a single user on a single link with capacity C. If the equilibrium queueing delay is larger than the maximum delay, $d^* > d_{max}$, then the system trajectory will hit the boundary $\mathcal{B} := \{(x, d) \in \mathbb{R}^2 : 0 \leq x \leq x_{max}, \text{ and } d = d_{max}\}$ at a point where $x > C$. We again implicitly assume that $x_{max} > C$. Since the vector field points toward $\mathcal{B}$, $\dot{d} > 0$, the trajectory cannot leave the boundary, resulting in a *sliding mode* behavior on the set $\{(x, d) \in \mathbb{R}^2 : C \leq x \leq x_{max}, \text{ and } d = d_{max}\}$. Then, the resulting vector field is a projection of the original system dynamics onto the boundary $\mathcal{B}$. A simplified sketch of the sliding mode behavior is shown in Figure 1.

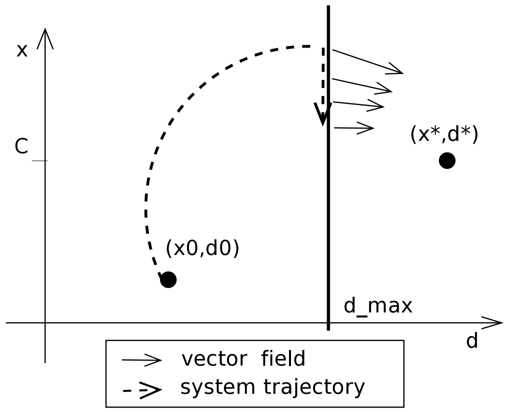


Fig. 1. Sliding mode behavior of a single user on a single link of capacity C.

For a general network topology, let the boundary $\mathcal{B}_g$ be defined as

$$\mathcal{B}_g := \{(\mathbf{x}, \mathbf{d}) \in \mathbb{R}^{M+L} \,:\, 0 \leq x_i \leq x_{i,max}, \text{ and } d_l = d_{l,max},\ \forall i,\, l\}$$

In the case the system trajectory hits $\mathcal{B}_g$, the set of users whose flows pass through the links l, where $d_l = d_{l,max}$, will experience packet losses. Let us define this set as

$$\mathcal{M}_{loss}(t) := \{i \in \mathcal{M} : \exists l \in R_i, \text{where } d_l(t) = d_{l,max},\ l \in \mathcal{L}\}. \tag{35}$$

Then, in the case of a packet loss, each user $i \in \mathcal{M}_{loss}$ dynamically increases $\alpha_i(t)$ multiplicatively, with a proportionality constant $\lambda > 0$. Otherwise, $\alpha_i(t)$ is decreased with a rate $1/\alpha_i(t)$ until the network-wide lower bound α_{min} is reached. Thus, the adaptive pricing scheme is defined by, $\forall i \in \mathcal{M}$,

$$\dot{\alpha}_i(t) = \begin{cases} \lambda\, \alpha_i(t)\ , \text{ if } (\mathbf{x}, \mathbf{d}) \in \mathcal{B}_g \text{ and } i \in \mathcal{M}_{loss}(t) \\ -\dfrac{1}{\alpha_i(t)}, \text{ if } \alpha_i > \alpha_{min}, (\mathbf{x}, \mathbf{d}) \notin \mathcal{B}_g, \text{ and } \\ \qquad\qquad\qquad\qquad i \notin \mathcal{M}_{loss}(t) \\ 0, \text{ if } \alpha_i = \alpha_{min},\, (\mathbf{x}, \mathbf{d}) \notin \mathcal{B}_g, \text{ and } i \notin \mathcal{M}_{loss}(t) \end{cases} \tag{36}$$

with system dynamics given in (15), and α_{min} being the network-wide lower bound on α. Global asymptotic stability of the system (15) was shown in Theorem III.7 for an inner equilibrium point. We now argue that the system is globally asymptotically stable under the adaptive pricing scheme (36). Consider the case when the system trajectory hits the boundary $\mathcal{B}_g$ and exhibits sliding mode behavior. Then, $\alpha(t)$ increases exponentially by (36) until the trajectory leaves the boundary, $\dot{\mathbf{d}} < 0$. Hence, we have a sequence of equilibria $(\mathbf{x}^*(\alpha), \mathbf{d}^*(\alpha))$ for each value of α. This has to happen in finite time, say t_0, since by (15) there exists an α' such that $\mathbf{x}'_l < C_l$ for all links l. Furthermore, at time t_0 the unique equilibrium of the system, $(\mathbf{x}^*(\alpha'), \mathbf{d}^*(\alpha'))$, becomes feasible by (19) as otherwise the trajectory cannot leave the boundary. For any feasible equilibrium point, we can define a sufficiently small level set, S_l, for the Lyapunov function V given by (22), which does not intersect with $\mathcal{B}_g$. Each hit of the system trajectory to $\mathcal{B}_g$ further shifts the equilibrium point by (19) and increases the size of the set S_l. Then, by global asymptotic stability of the system (see Theorem III.7) the trajectory has to enter this invariant level set in finite time $t_1 \geq t_0$, and converge asymptotically to the unique equilibrium point. Thus, we have the following theorem summarizing this result:

Theorem V.1. *Let $\mathbf{A}$ be full row rank, and let the boundary $\mathcal{B}_g$ be given by $\mathcal{B}_g := \{(\mathbf{x}, \mathbf{d}) \in \mathbb{R}^{M+L} : 0 \leq x_i \leq x_{i,max}$, and $d_l = d_{l,max},\ \forall i,\, l\}$ due to the limited buffer sizes at the nodes of the network. Let the set of users experiencing packet losses be given by (35). Define the adaptive pricing scheme as*

$$\dot{\alpha}_i(t) = \begin{cases} \lambda\, \alpha_i(t)\ , \text{ if } (\mathbf{x}, \mathbf{d}) \in \mathcal{B}_g \text{ and } i \in \mathcal{M}_{loss}(t) \\ -\dfrac{1}{\alpha_i(t)}, \text{ if } \alpha_i > \alpha_{min}, (\mathbf{x}, \mathbf{d}) \notin \mathcal{B}_g, \text{ and } \\ \qquad\qquad\qquad\qquad i \notin \mathcal{M}_{loss}(t) \\ 0, \text{ if } \alpha_i = \alpha_{min},\, (\mathbf{x}, \mathbf{d}) \notin \mathcal{B}_g, \text{ and } i \notin \mathcal{M}_{loss}(t) \end{cases}$$

for all $i \in \mathcal{M}$. If $(\mathbf{x}^*(\alpha), \mathbf{d}^*(\alpha)) \in \mathcal{B}_g$ then there exists a finite time t_0 such that the sequence of unique equilibria $(\mathbf{x}^*(\alpha), \mathbf{d}^*(\alpha))$ of the system (15), indexed by α, converges to a feasible inner solution $(\mathbf{x}^*_{inner}, \mathbf{d}^*_{inner}) \in \Omega$, where Ω is as defined in (26). Furthermore, the sequence of equilibria $(\mathbf{x}^*(\alpha), \mathbf{d}^*(\alpha))$ cannot leave the feasible set Ω except for a finite time.

We note that the Theorem V.1 can be straightforwardly extended to capture the single bottleneck system with propagation delays described in (31).

Remark V.2. The convergence of the system to a feasible equilibrium point with adaptive pricing scheme occurs in two different time scales. In the fast time scale, the vector $\alpha(t)$ converges through the dynamics given by (36) to a specific value which is associated with an inner equilibrium point of Ω. With the value of α very slowly changing as described, the system (15) then asymptotically converges in the slower time scale to this unique inner equilibrium point. Since the implementation of the adaptation algorithm and the system will be in discrete-time, relocation of the equilibrium and increase in the pricing parameter α will occur in discrete jumps separated by RTT instead of a continuous shift. □

Remark V.3. Under the adaptive pricing scheme introduced individual users are charged according to the congestion levels on their current path. If some of the links on the path of the user are very congested than this leads to packet losses and higher prices (36). This is similar to existing pricing mechanisms such as peak hours versus night and weekends in current telecommunication (voice) networks. We also note that due to the bursty nature of packet losses on the links, approximately all of the users using that link experience the packet loss event that leads to a fair price adjustment. This conjecture is supported by the simulation results observed in Section VII (see Figure 6(a)). In addition, the dynamic nature of (36) enables the prices adapt to variations in congestion levels, resulting in a more efficient and fair bandwidth allocation.

VI. An Implementation of the Congestion Control Scheme

The user responses in Section II are based on a continuous time formulation. In reality, however, users update their flow rates only at discrete time instances corresponding to multiples of RTT. Hence, for implementation purposes, we discretize the reaction function of the i^{th} user, and normalize it with respect to the RTT of the user. In addition, we need a specific utility function in order to quantify the user response in (6). Logarithmic utility functions are widely used in the literature not only because they have nice properties like strict concavity but also because they adequately capture several important concepts from economics, such as the law of diminishing returns. We choose the following utility function for i^{th} user:

$$U_i(x_i) = u_i \log(x_i + 1),$$

where u_i is a user specific utility parameter. The optimal user response is, therefore, a discretized version of (6), and is given by

$$x_i(t+1) = \left[x_i(t) + \kappa_i \Big[\frac{u_i}{x_i(t) + 1} - \alpha_i \sum_{l \in R_i} d_l(t) \Big] \right]^+ , \quad (37)$$

where κ_i is a (user specific) step-size constant. The adaptive pricing scheme (described in Section V) for the i^{th} user is given by (again in the discrete-time domain)

$$\alpha_i(t+1) = \begin{cases} \lambda\, \alpha_i(t), & \text{if a packet loss occurs} \\ -\dfrac{base\_RTT}{\alpha_i(t)}, & \text{if } \alpha_i > \alpha_{min} \text{ and no packet loss} \\ \alpha_i(t), & \text{if } \alpha_i = \alpha_{min} \text{ and no packet loss} \end{cases} \quad (38)$$

where λ is chosen as 1.20, and $base\_RTT$ is the minimum RTT experienced by the user.

Remark VI.1. In the actual implementation, the proposed decentralized pricing mechanism, including the update of α_i and calculation of d_i, for the i^{th} user has to be performed by a closed (software) module at the user node, which cannot be manipulated by the user. Then, prices in terms of network credits can be related to real world prices. Alternatively, it is possible to implement the whole cost structure on a voluntary basis by the users as in the case of TCP. □

The congestion control scheme characterized by the user response (37) is implemented in a Game (theory) Based Congestion Control (GBCC) protocol using the Network Simulator 2 (*ns-2*) [35]. The simulator *ns-2* is chosen because it provides both a realistic environment for testing the proposed congestion control scheme and a level of abstraction for easy implementation. GBCC is a simple window based protocol for best-effort data traffic. It is devised as an end-to-end sliding window protocol [36], where the sender side adjusts its window size according to the reaction function (37). For simplicity, receiver window size is chosen as one. We also implement a version with a simple slow start mechanism where the window size is increased by one per RTT until a packet loss is observed. The GBCC scheme is then extended to Adaptive GBCC (AGBCC) using the adaptive pricing (38). We next provide an overview on the GBCC and AGBCC schemes by summarizing the sender and receiver side functionalities.

A. GBCC Protocol

As one of the goals of GBCC protocol is compatibility with existing protocols, most of the functionality is on the sender side. Specifically, the sender side has the following functions:

- The sender puts sequence number and time stamp into the packet header. It estimates RTT and base RTT, where the latter is calculated as the minimum of the RTTs up to that point, by using the received acknowledgment (ack) packets. The estimation method for RTT is the same as the one in [37].
- If a double ack is received, i.e. the same packet is acknowledged twice by the receiver, then it retransmits the packages beginning from the last acknowledged packet number. We note that this *go back n* scheme [36] is implemented for its simplicity. In fact, better mechanisms with receiver window size being larger than 1 do exist.
- The sender updates the window size according to (37) using the current value of queueing delay, which is taken as the difference between the current RTT and the base RTT. The window size, W, is strictly positive.
- If no ack packet is received within, say $2\,RTT$, then sender retransmits previous packets beginning from the last acknowledged one, and reduces the window size.

The receiver side, on the other hand, has the function of acknowledging received packets. If no packet is received for a specific time, say $4\,RTT$, the last received packet is acknowledged again.

B. AGBCC Protocol

In the AGBCC protocol, sender and receiver have the same functionality as in GBCC. However, in addition to standard functions, each sender also adjusts its own pricing parameter α at each packet loss in accordance with the adaptive pricing scheme (38). The change of α is proportional to base RTT if there are no packet losses and $\alpha > \alpha_{min}$ in order to equalize the rate of change among users with different delays.

VII. Simulations

We simulate the proposed congestion control schemes, GBCC and AGBCC, on *ns-2*. The underlying protocol used for routing is the standard IP. Links and queues are chosen to be duplex and drop-tail, respectively. For simplicity, we fix the packet sizes to $1,000\ bytes$. In most of the cases queueing delays on the links are much smaller than the propagation delays that we choose. Hence, RTT's are approximately equal to twice the propagation delays.

First, we simulate GBCC without a slow start mechanism in the simple single-user single-link case. The parameters in (37) are chosen as $\alpha = 30$ and $u = 10,000$. The buffer size is $50KB$ and propagation delay on the link is varied from $5ms$ to $25ms$, and to $100ms$. We observe in Figure 2(a) that as RTT gets too large, the system becomes unstable in accordance with the analysis in Section IV. Notice that it takes up to 7 seconds

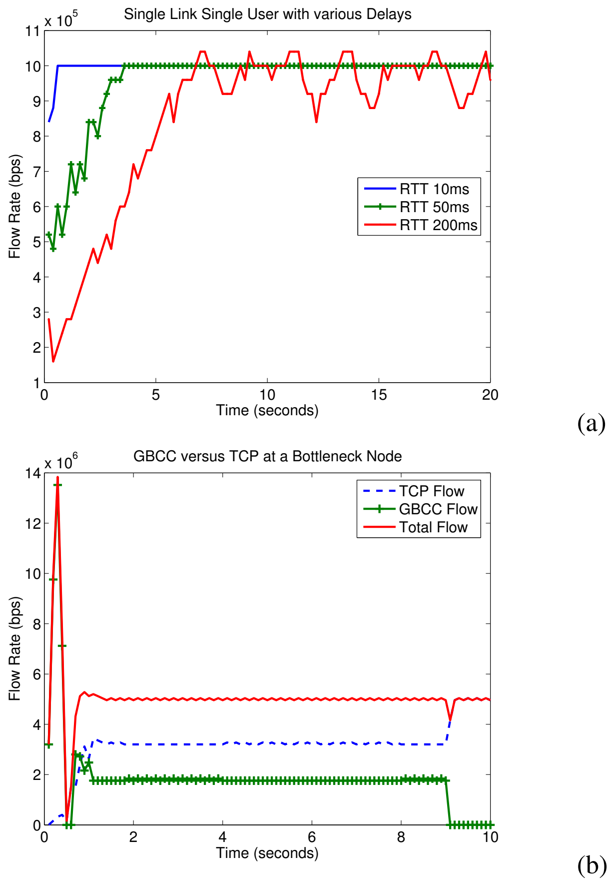


Fig. 2. A single user on a single link with $RTT = 10, 50, \text{and}\, 200ms$ (a). GBCC flow versus TCP flow on a bottleneck link with $10ms$ propagation delay (b).

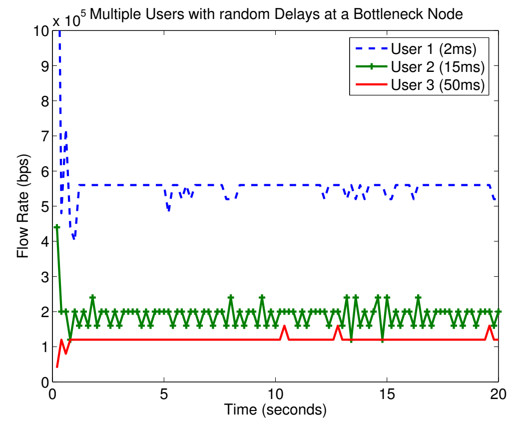


Fig. 3. Three out of 20 flows with various propagation delays ($2ms$, $15ms$, and $50ms$) sharing a $5Mbps$ bottleneck link.

for the flow to reach its capacity in this simulation. Therefore, we use the slow start version of GBCC for the rest of the study.

We next explore the interaction between GBCC and TCP on a single bottleneck link with $10ms$ propagation delay. GBCC is TCP-friendly [16] since its long-term rate does not exceed the one of the TCP flow as observed in Figure 2(b). The fluctuation in the first two seconds is due to the slow start mechanism which requires a packet loss for termination. In the final simulation on a single bottleneck link, there are 20 identical users with parameters $\alpha = 50$, $u = 400,000$, and propagation delays are randomly chosen between $2ms$ and $50ms$ according to a uniform distribution. We observe flows of 3 specific users with respective propagation delays of $2ms$, $15ms$, and $50ms$ in Figure 3. The system again converges to the equilibrium, however similar to TCP, GBCC favors flows with smaller RTT as it is a window-based scheme.

We then carry out a simulation with three users on a simple three node network topology with two $5Mbps$ links of $20ms$ propagation delay as shown in Figure 4(a). While flows of users 1 and 2 pass through links 1 and 2 respectively, the flow of user 3 passes through both links. Cost parameters are chosen as $\alpha = 30$ and $u = 400,000$. User 3 is 'charged' more than others through summation of queueing delays as s/he uses resources on both links. Thus, having the same utility parameter as others, s/he obtains a smaller fraction of the bandwidth. Figure 5(a) depicts the flow rates of users 2 and 3, as observed in node 2.

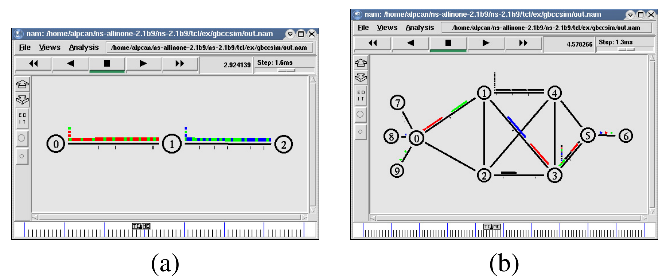


Fig. 4. *Nam* screenshots of the simple network (a) and of the general (arbitrary) topology network.

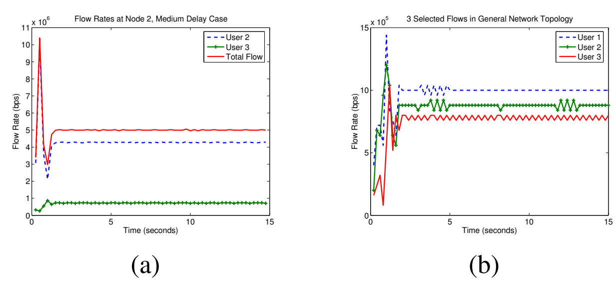
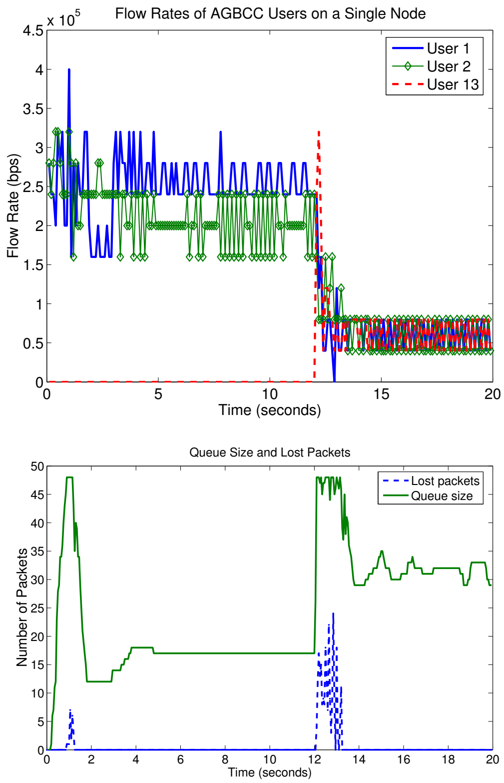


Fig. 5. Flows of users 2, 3, and total flow at node 2 are observed for 15 seconds (a). Three symmetric flows from nodes 7, 8, and 9 to node 6 (b).

We next simulate 10 users with various routes and experiencing various information delays on a seven node arbitrary topology network (Figure 4(b)) with all links except the one between nodes 5 and 6 having capacity of $5Mbps$ each. The link between nodes 5 and 6, on the other hand, has a capacity of $10Mbps$. The links have equal propagation delays of $5ms$ each, except the links to nodes 7, 8, and 9, which have delays of $5ms$, $10ms$, and $25ms$, respectively. The users at nodes 7, 8, and 9 all have connections to node 6 and each experiences a different propagation delay. Figure 5(b) shows only the flows of these three users as measured at node 6. We note that although the number of links in this simulation is equal to the number of users, the number of bottleneck links that affect the equilibrium flows is actually smaller. Hence, the routing matrix $\mathbf{A}$ is of full row rank.



(a)

(b)

Fig. 6. The flow rates of 3 selected AGBCC users out of 20 on a single bottleneck link (a). Queue size and the number of packets lost at the bottleneck link (b).

The adaptive pricing scheme studied in Section V is simulated with 20 AGBCC users on a single bottleneck link of $5Mbps$ capacity. The parameters in (37) are chosen as $\alpha = 30$ and $u = 30,000$. The users are divided into two groups of 10. First group of users start transmitting immediately, whereas the second group starts transmission at $t = 12s$. The flow rates of three selected users two of which belonging to the first group are shown in Figure 6(a). We also observe in Figure 6(b) the number of packets in the the queue and the number of lost packets at the bottleneck link. Due to the packet losses in the beginning of the simulation as well as at $t = 12s$ pricing parameter is increased, and hence, the operating point of the system is shifted. Finally, we simulate 10 AGBCC users on the arbitrary topology network of Figure 4(b) with the same set of parameters as before. The results are consistent with the ones of the single bottleneck case, and show that the adaptive pricing scheme functions in accordance with Theorem V.1.

VIII. Concluding Remarks

In this paper, we have developed and analyzed a congestion control game with a linear pricing scheme based on variations in the queueing delay experienced by the users. User demand for bandwidth is captured by a broad class of utility functions that are strictly increasing and strictly concave. The objective function for each user in this noncooperative game is defined as the difference between the pricing and utility functions. Using a network model based on fluid approximations, and through a realistic modeling of queues in the network, we have established the existence of a unique equilibrium, and the global stability of the equilibrium point for a general network topology. We have also provided sufficient conditions for system stability on a bottleneck link shared by multiple users under non-negligible information delays. In addition, we have studied an adaptive pricing scheme for adjusting the pricing parameter dynamically in order to increase flexibility and robustness of the congestion control game.

We have implemented and simulated a simple, window-based, end-to-end congestion control scheme in *ns-2* network simulator based on the theoretical foundations of the congestion control game. We have investigated several properties of the scheme developed through simulations on a single bottleneck link and on various general network topologies with non-negligible propagation delays. These simulations reveal that the implemented scheme not only confirms the theoretical results but is also TCP-friendly.

There still remain a number of open issues and many directions for future research. For example, there is still ample room for improvement in the implementation of the congestion control scheme, such as increasing the receiver window size and fine tuning the slow start mechanism. Another topic for further study would be the derivation of improved (less restrictive) sufficient conditions on the maximum delay allowable in a general network, to ensure stability of the overall system.

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