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A STABILITY RESULT FOR SWITCHED SYSTEMS WITH MULTIPLE EQUILIBRIA

Tansu Alpcan[1] and Tamer Başar[2]

[1]Deutsche Telekom Laboratories
Technical University of Berlin, Germany.
email: alpcan@sec.t-labs.tu-berlin.de

[2]Coordinated Science Laboratory, University of Illinois
1308 West Main Street, Urbana, IL 61801 USA.
Corresponding author email: basar1@illinois.edu

Abstract.

This paper studies stability properties of general switched systems with multiple distinct equilibria. It is shown that, if the dwell time of the switching events is greater than a certain lower bound, then the trajectory of a general switched system with multiple distinct equilibria, where each system is exponentially stable, globally converges to a superset of those equilibria and remains in that superset.

Keywords. Nonlinear systems, switched systems, dwell time, Lyapunov theory, stability analysis.

AMS (MOS) subject classification: 34K34-Hybrid systems, 93D05-Lyapunov and other classical stabilities.

1 Introduction

A switched system consists of a family of continuous-time dynamical (sub)systems and a switching rule or signal that governs the switching between them [9]. Switched systems, which are closely related to hybrid systems, are encountered in many applications ranging from mechanical and power systems to automotive and aerospace industries. Consequently, switched/hybrid systems theory has become an important subfield of systems and control theory in recent years. Applications of switched systems approach include study of power control in wireless networks [2], robustness of congestion control schemes in wired networks [1], and dynamic bipedal locomotion [4].

Heretofore the stability analysis of switched systems has generally been restricted to a single equilibrium point common to all subsystems. Stability results within such a framework have been reported in [5, 8, 9]. However, these results do

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[1]T. Alpcan was with University of Illinois during part of this research.

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not apply to the more general framework where each subsystem has its own (distinct) equilibrium point. Such cases arise, for example, in game theoretical models where each switching event corresponds to a different game, with each such game having a unique Nash equilibrium [3]. Then, the equilibrium point of the overall game shifts with each switching event.

Accordingly, we investigate in this paper stability of general switched systems with multiple (distinct) equilibria such as the ones in [1, 2, 4]. An application-independent presentation of these stability results should be of interest to the hybrid systems community. Since one cannot expect any single equilibrium point to be globally attractive, the next best thing would be for the trajectory of the switched system to converge to a some set containing all the equilibria. We make this property precise in this paper, and show that with proper formulation there is such a set of attraction. The next section provides some basic definitions and preliminaries. Section 3 presents the main result, which is an extended stability result for switched systems with multiple equilibria, and Section 4 includes two numerical examples which illustrate the result. The paper ends with the concluding remarks of Section 5.

2 Definitions and Preliminaries

Consider a continuous-time dynamical system, labeled q, with state $\mathbf{x}$ taking values in the D-dimensional Euclidean space $X = \mathbb{R}^D$. The state evolves for each q according to

$$\dot{\mathbf{x}} = F^{(q)}(\mathbf{x})\,, \tag{1}$$

which has a unique equilibrium point $\mathbf{x}^{*(q)} \in X$. Define a family (set) of such subsystems[1]

$$Q := \{q_1, q_2, \ldots, q_{max}\}.$$

We refer to a (discrete) switch or jump from one such subsystem $q \in Q$ to another subsystem $r \in Q$ as a *switching* event. A switching signal $\sigma(t)$ is defined on (t_0, ∞) as a piecewise constant function that takes its values from Q, and has a finite number of discontinuities (switches) on any finite subinterval. The switching signal determines when the system switches from one subsystem to another, and hence the specific subsystem dynamics according to which the system trajectory will evolve at any given time $t \in (t_0, \infty)$. The family Q of subsystems under such a switching scheme constitutes a *switched system* [8], which is sometimes referred to also as a *hybrid system*.

In the study of switched systems of the type described above, we make use of the concept of *dwell time*, τ, which quantifies the minimum amount of time elapsed between two switches [5, 8, 9]. Let us denote the number of discontinuities of a switching signal $\sigma(t)$ when restricted to a finite interval $[t_0, T]$ by $N_\sigma(t_0, T)$. An

[1] The set Q introduced below refers by a slight abuse of notation to both the labeling of the subsystems as well as the collection of the subsystems themselves.

example switching signal might be, when restricted to a finite interval $[t_0, T)$,

$$\sigma(t) = \begin{cases} q_1 & \text{if } t \in (t_0, t_1^-) \\ q_2 & \text{if } t \in (t_1^+, t_2^-) \\ q_1 & \text{if } t \in (t_2^+, t_3^-) \\ q_3 & \text{if } t \in (t_3^+, T), \end{cases}$$

where $q_1, q_2, q_3 \in Q$ and $t_0 < t_1 < t_2 < t_3 < T$. The dwell time of such a signal would be no smaller than the quantity $\min(t_1 - t_0,\, t_2 - t_1,\, t_3 - t_2,\, T - t_3)$.[2]

For future reference, let us first recall the main Lyapunov-based stability result for *non-switching* dynamic systems, captured as a special case of our formulation here, with $q \in Q$ fixed and holding for all t. Suppose that there exist $\mathcal{C}^1$ functions $V^{(q)} : X \to \mathbb{R}$, $q \in Q$, and class $\mathcal{K}_\infty$ functions[3] χ_1 and χ_2 such that [4]

$$\chi_1(||\mathbf{x} - \mathbf{x}^{*(q)}||) \leq V^{(q)}(\mathbf{x}) \leq \chi_2(||\mathbf{x} - \mathbf{x}^{*(q)}||),\ \forall q \in Q, \tag{2}$$

and further there exists a positive number ε such that

$$\dot{V}^{(q)}(\mathbf{x}) \leq -\varepsilon V^{(q)}(\mathbf{x}),\ \forall \mathbf{x} \in X,\ \forall q \in Q. \tag{3}$$

Then, the unique equilibrium point $\mathbf{x}^{*(q)}$ of each subsystem $q \in Q$ is globally exponentially stable by the Lyapunov stability theorem [7].

Now, any extension of this result to set convergence requires the introduction of a common point-to-set distance

$$d(\mathbf{x}, \mathcal{A}) := \inf_{\eta \in \mathcal{A}} ||\mathbf{x} - \eta||$$

for each nonempty set $\mathcal{A} \subset X$ and point $\mathbf{x} \in X$ [10]. For a given class of switching signals (characterized for example by a dwell time constraint), the system (1) is said to *globally converge* to a set $\mathcal{A}$, if given any $\delta > 0$, there exists a finite $\bar{t} > t_0$ such that for any switching signal $\sigma(t)$ from the given class,

$$d(\mathbf{x}(t_0), \mathcal{A}) \leq \delta \quad \to \quad \mathbf{x}(t) \in \mathcal{A}\ \ \forall t > \bar{t}. \tag{4}$$

We next introduce two subsets of X which will play an important role in the main result of this paper. The first is the union of some neighborhoods of each equilibrium point, with the neighborhoods defined in terms of the individual Lyapunov functions introduced above. This first set is not necessarily connected. The second one is a superset that contains this union but is connected. Now, for a mathematically precise description, let κ be a positive constant. Introduce $\mathcal{N}^{(q)}(\kappa)$ as a set (closed neighborhood) around the equilibrium $x^{*(q)}$ of a subsystem $q \in Q$:

$$\mathcal{N}^{(q)}(\kappa) := \left\{\mathbf{x} \in X : V^{(q)}(\mathbf{x}) \leq \kappa\right\}, \tag{5}$$

[2] It is *no smaller* because we have not specified what the switching times are outside $[t_0, T)$.

[3] A function $f(\cdot) : \mathbb{R}^+ \to \mathbb{R}^+$ is a $\mathcal{K}_\infty$ function, if it is continuous, strictly increasing, $f(0) = 0$, and $f(t) \to \infty$ as $t \to \infty$.

[4] $||\cdot||$ below is a standard norm in a finite dimensional space such as the Euclidean norm.

and further let its union over Q characterize a specific superset of the equilibria of system (1):

$$\mathcal{N}(\kappa) := \bigcup_{q \in Q} \mathcal{N}^{(q)}(\kappa)\,, \tag{6}$$

which, as indicated earlier, is not necessarily a connected set. The second set we introduce, $\mathcal{L}(\kappa)$, is defined as follows: Let

$$\alpha^{(q)}(\kappa) := \max_{\mathbf{x} \in \mathcal{N}(\kappa)} V^{(q)}(\mathbf{x}),$$

and

$$\mathcal{M}^{(q)}(\kappa) := \{\mathbf{x} : V^{(q)}(\mathbf{x}) \le \alpha^{(q)}(\kappa)\},$$

for each $q \in Q$. Then,

$$\mathcal{L}(\kappa) := \bigcup_{q \in Q} \mathcal{M}^{(q)}(\kappa). \tag{7}$$

In other words, the set $\mathcal{L}(\kappa)$ is the union of the closed sublevel sets of $V^{(q)},\ \forall q \in Q$, that contain a superset of equilibria $\mathcal{N}(\kappa)$ given the positive constant κ. By definition,[5] $\mathcal{L}(\kappa)$ is a connected set as there is at least one overlapping equilibrium point between each pair of sets $(\mathcal{M}^{(i)}, \mathcal{M}^{(j)}),\ 1 \le (i,j) \le q_{max}$. In addition, note that $\mathcal{N}(\kappa) \subset \bigcap_{q \in Q} \mathcal{M}^{(q)}$ by definition.

3 The Main Result

The stability of switching systems with a single equilibrium point, that is with $\mathbf{x}_q^* = \mathbf{x}_r^*,\ \forall q, r \in Q$ in our formulation, has been studied extensively in the literature (see, for example, Theorem 3.2 of [8] and Theorem 2 of [5, 6], which establish global asymptotic stability). However, these results do not apply to our framework, as the equilibrium point of one subsystem may not be the same as the equilibrium point of another subsystem. Since at the time of switching the system equilibrium shifts from one point to another, no single point in the system can be asymptotically stable. In a sense, the special sets $\mathcal{N}(\kappa)$ and $\mathcal{L}(\kappa)$ we introduced in the previous section help extend the single equilibrium point concept, and thus the result to be presented next constitutes a generalization of the existing results to the multiple equilibrium case.

The main result of this paper is now given below, where we make use of the definitions and concepts introduced in Section 2.

Theorem 1. *Consider a family of systems Q as defined by (1), each with a unique equilibrium point, $\mathbf{x}^{*(q)}$. Suppose that there exist $\mathcal{C}^1$ functions $V^{(q)} : X \to \mathbb{R}$, $q \in Q$, class $\mathcal{K}_\infty$ functions χ_1 and χ_2, and a positive number ε such that (2) and (3) hold. Furthermore, let $\mathcal{N}(\kappa)$ and $\mathcal{L}(\kappa)$ be as defined by (6) and (7), respectively, for a given positive constant κ. Introduce $\mu(\kappa) \in (1, \infty)$ such that*

$$\frac{V^{(q)}(\mathbf{x})}{V^{(r)}(\mathbf{x})} \le \mu(\kappa), \quad q, r \in Q,\ \forall \mathbf{x} \in X - \mathcal{N}(\kappa)\,. \tag{8}$$

[5] Recall that the Lyapunov functions here are assumed to be $\mathcal{C}^1$.

Then, for every switching signal $\sigma(t)$ with dwell time τ satisfying

$$\tau > \frac{\log \mu(\kappa)}{\varepsilon}, \tag{9}$$

the state trajectory of the switched system globally converges to the set $\mathcal{L}(\kappa)$.

Proof. Let us consider an arbitrary but fixed switching signal $\sigma(t),\ t \in [t_0, \infty)$ with switching times $\{t_1, \ldots, t_i, \ldots\}$. Now consider its restriction to a finite time interval $[t_0, T]$ with corresponding switching times $S = \{t_1, \ldots, t_{N_\sigma}\}$, where N_σ is the number of switches in this interval, and $t_0 < t_1 < t_{N_\sigma} < T$. Let $t_0 := 0$, and initially assume that $\mathbf{x}(0) \notin \mathcal{N}(\kappa)$. In this part of the proof we further assume that $\mathbf{x}(t)$ does not enter $\mathcal{N}(\kappa)$ during the time interval $[t_0, T]$. If this happens, then the second part of the proof will apply.

Given the switching signal $\sigma(t)$ on this interval $[0, T]$, define the piecewise continuously differentiable function $W(t)$ as

$$W(t) := e^{\varepsilon t} V^{(\sigma(t))},$$

where $V^{(\sigma(t))}$ denotes the Lyapunov function of the "current" subsystem $\sigma(t) \in Q$, and ε is as introduced in (3).

On each subinterval $[t_i, t_{i+1})$ (between two consecutive switching times t_i and t_{i+1}), we have from (3),

$$\dot{W}(t) = \varepsilon W(t) + e^{\varepsilon t} \dot{V}^{(\sigma(t))} \leq \varepsilon W(t) - e^{\varepsilon t} \varepsilon V^{(\sigma(t))} = 0.$$

This leads to the inequality $W(t_{i+1}^-) \leq W(t_i)$. In addition, by (8), $W(t_{i+1}^+) \leq \mu W(t_{i+1}^-)$. Hence, following a switching event, we have

$$W(t_{i+1}^+) \leq \mu W(t_{i+1}^-) \leq \mu W(t_i).$$

Repeating this process for all switching events $i = 1, \ldots, N_\sigma$, we arrive at

$$W(T^-) \leq W(t_{N_\sigma}) \leq \mu^{N_\sigma} W(0)\,.$$

Multiplying both sides by $e^{-\varepsilon T^-}$, and using the definition of W,

$$V^{(\sigma(T^-))}(\mathbf{x}(T^-)) \leq e^{-\varepsilon T^-} \mu^{N_\sigma} V^{(\sigma(0))}(\mathbf{x}(0)).$$

Then, from (9) and $N_\sigma \leq T/\tau$, we have

$$V^{(\sigma(T^-))}(\mathbf{x}(T^-)) \leq e^{((\log \mu(\kappa)/\tau) - \varepsilon) T^-}\ V^{(\sigma(0))}(\mathbf{x}(0)). \tag{10}$$

Note that from (9), we have

$$1 > e^{((\log \mu(\kappa)/\tau) - \varepsilon) T^-} \ \text{ and } \lim_{T^- \to \infty} e^{((\log \mu(\kappa)/\tau) - \varepsilon) T^-} = 0.$$

The above then says that for a given initial state $\mathbf{x}(0)$, and with any switching signal of the type specified in the theorem, the right-hand side of (10) can be made arbitrarily small by taking T sufficiently large. Equivalently, for the problem defined on $[0, \infty)$, there exists an integer i such that the state trajectory enters the set $\mathcal{N}^{(\sigma(t_i))}(\kappa)$ within the interval $[t_i, t_{i+1})$. Hence the trajectory enters $\mathcal{N}(\kappa)$ in finite time (but not stay in this set, unless there are no subsequent switches). We will next show in the second part of the proof that once the trajectory enters $\mathcal{N}(\kappa)$, then it never leaves $\mathcal{L}(\kappa)$.

Note that, even if the trajectory starts in $\mathcal{L}(\kappa) - \mathcal{N}(k)$, then it does not have stay within $\mathcal{L}(\kappa)$ initially. However, the analysis above indicates that there exists a finite time instance after which it enters the set $\mathcal{N}(\kappa)$. This finite time instance, say $\bar{t}$, depends on the initial state $x(0)$, dwell time τ, and the quantities μ and ϵ introduced earlier.

We next show that if at some time point $t_1 \geq 0$ the trajectory is in $\mathcal{N}(\kappa)$ (or starts in there), then it stays in $\mathcal{L}(\kappa)$ for all $t > t_1$. Consider the restriction of the switching signal $\sigma(t)$ to the finite time interval $[t_1, T_1]$ with switching times on that interval $\tilde{S} = \{t_2, \ldots, t_{N_\sigma}\}$ where $0 < t_2 < t_{N_\sigma} < T_1$. Assume first that $x(t_1) \in \mathcal{N}(\kappa)$ to be near the equilibrium of the current subsystem $x^{*(r)}$, where $r = \sigma(t_1)$, such that $V^{(r)}(x(t_1)) \leq \kappa$, that is $x(t_1) \in \mathcal{N}^{(r)}(\kappa)$. Then, by definition we have $\mathbf{x}(t) \in \mathcal{N}(\kappa)$ for $t \in [t_1, t_2)$, i.e. until the first switching event. Let us assume a worst-case scenario,[6] where the first switching event is such that $\sigma(t_2^+) \neq r$ and the interval $t_3 - t_2$ is long enough such that the trajectory moves outside $\mathcal{N}(k)$. Then, the inequality $W(t_3^-) \leq W(t_2^+) \leq \mu W(t_2^-)$ holds. Assumption (8) and definition of W lead to

$$V^{(\sigma(t_3^-))}(\mathbf{x}(t_3^-)) \leq e^{\log \mu(\kappa) - \varepsilon \Delta} \; V^{(\sigma(t_2^-))}(\mathbf{x}(t_2^-)),$$

where $\Delta := t_3 - t_2 \geq \tau$ by definition of dwell time. From

$$V^{(\sigma(t_3^-))}(\mathbf{x}(t_3^-)) \leq e^{\log \mu(\kappa) - \varepsilon \Delta} \; V^{(\sigma)}(\mathbf{x}(t_2)) \leq e^{\log \mu(\kappa) - \varepsilon \tau} \; V^{(\sigma(t_2^-))}(\mathbf{x}(t_2^-)),$$

condition (8), and definition of $\mathcal{N}(k)$, it directly follows that

$$V^{(\sigma(t_3^-))}(\mathbf{x}(t_3^-)) \leq V^{(\sigma(t_2^-))}(\mathbf{x}(t_2^-)) \leq \kappa.$$

Therefore, $\mathbf{x}(t_3^-) \in \mathcal{N}(\kappa) \subset \mathcal{L}(\kappa)$ or the trajectory necessarily returns to the set $\mathcal{N}(\kappa)$ before the next switching event, albeit near the equilibrium point of another subsystem. Under the switching signal $q_1 = \sigma(t)$ in the time interval $t_1 \leq t < t_2$, the trajectory is in $\mathcal{M}^{(q_1)}(\kappa)$. Likewise, it is in $\mathcal{M}^{(q_2)}(\kappa)$ under the switching signal $q_2 = \sigma(t), \; t_2 \leq t < t_3$. Since both $\mathcal{M}^{(q_1)}(\kappa) \subset \mathcal{L}(\kappa)$ and $\mathcal{M}^{(q_2)}(\kappa) \subset \mathcal{L}(\kappa)$ hold by definition of $\mathcal{L}$, the trajectory $\mathbf{x}(t)$ stays in $\mathcal{L}(\kappa)$ during the time interval $[t_1, t_3)$.

We now consider the case when $x(t_1) \in \mathcal{N}(\kappa)$ is not near the equilibrium of the current subsystem $x^{*(r)}$, i.e. for example $x(t_1) \in \mathcal{N}^{(m)}(\kappa)$, where $m \neq r$. In this case, unlike the analysis above, the trajectory exits[7] $\mathcal{N}(k)$, which then leads to the same situation as when $x \in \mathcal{L}(\kappa) - \mathcal{N}(k)$. □

[6] The special cases where the trajectory stays near the initial equilibrium as a result of a special switching sequence which does not force it outside do not essentially affect the argument. We can always define t_2 as the time of the first switch that leads to the worst-cases scenario.

[7] There may be special cases such as $\mathcal{N}(k)$ being connected and a specific switching signal resulting in the case above where the trajectory is near the equilibrium of the subsystem corresponding to the signal. These cases have already been covered.

Corollary 2. *Under the assumptions of Theorem 1, for all starting points* $\mathbf{x}(0) \in \mathcal{N}(\kappa)$*, the switched system remains in the set* $\mathcal{L}(\kappa)$*, i.e.*

$$\mathbf{x}^{(\sigma(t))}(t) \in \mathcal{L}(\kappa), \ \ \forall t > 0,$$

for every switching signal $\sigma(t)$ *with dwell time* $\tau > \log \mu(\kappa)/\varepsilon$*.*

It is interesting to note that in this multi-equilibrium system, the convergence to the target set $\mathcal{L}$ may not happen necessarily when the trajectory first enters this set. It only happens when the trajectory enters the smaller subset $\mathcal{N}$. In other words, the $\bar{t}$ mentioned in the proof of the theorem above, is determined through entrance to $\mathcal{N}$, and not $\mathcal{L}$. Only then (for $t > \bar{t}$) the trajectory necessarily remains in $\mathcal{L}$.

We also note that the condition of Theorem 1 is only sufficient, because the proof has used a worst case analysis. It might be possible for the trajectory to converge to a smaller set (than $\mathcal{L}(\kappa)$, if more structure is brought to the system or the switching signal. Also, it would be useful to determine the smallest possible value of κ for which the result of the theorem holds. Note that if there were only one equilibrium state (the classical case), then κ can be taken to be arbitrarily small, and μ can be taken to be a constant. The result of the theorem is then in the spirit of the available results for the single equilibrium case, such as Theorem 3.2 of [8], where global asymptotic stability to the unique equilibrium state is assured if the *average* dwell time is larger than $\log \mu/\epsilon$.

4 Numerical Examples

We provide two numerical examples to illustrate the main result presented in the previous section. We note that in both examples convergence to the equilibrium set is in finite time as expected. Further, both examples demonstrate that the sufficient condition of Theorem 1 is fairly conservative, which can be attributed to its general nature.

4.1 Example 1

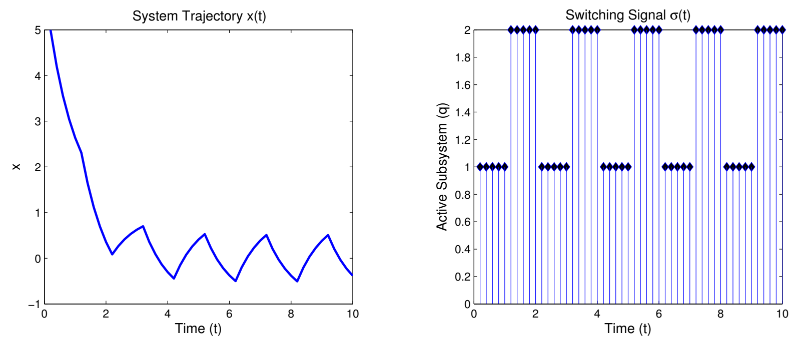
Let $X = \mathbb{R}$, $Q = \{1, 2\}$, $F^{(1)}(x) = -x + 1$, and $F^{(2)}(x) = -x - 1$. Define the corresponding Lyapunov functions as $V^{(1)}(x) := (x-1)^2$ and $V^{(2)}(x) := (x+1)^2$, respectively. Then, the ε parameter in (3) is $\varepsilon = 2$.

From definitions in Section 2, $\mathcal{N}(\kappa) = [1-\sqrt{\kappa}, 1+\sqrt{\kappa}] \cup [-1-\sqrt{\kappa}, -1+\sqrt{\kappa}]$, $\alpha^{(1)}(\kappa) = \alpha^{(2)}(\kappa) = (2+\sqrt{\kappa})^2$, $\mathcal{M}^{(2)}(\kappa) = [-1-\sqrt{\kappa}, 3+\sqrt{\kappa}]$, and $\mathcal{M}^{(1)}(\kappa) = [-3-\sqrt{\kappa}, 1+\sqrt{\kappa}]$. Thus, $\mathcal{L}(\kappa) = [-3-\sqrt{\kappa}, 3+\sqrt{\kappa}]$. Then,

$$\mu(\kappa) = \max_{\mathbf{x} \notin \mathcal{N}(\kappa)} \frac{V^{(1)}(\mathbf{x})}{V^{(2)}(\mathbf{x})} = \max_{\mathbf{x} \notin \mathcal{N}(\kappa)} \frac{V^{(2)}(\mathbf{x})}{V^{(1)}(\mathbf{x})} = \frac{(4+\sqrt{\kappa})^2}{(2+\sqrt{\kappa})^2},$$

For $\kappa = 0.01$, we obtain $\mu = 3.81$ and $\mathcal{L} = [-3.1, 3.1]$. Hence, the lower bound on the dwell time is $\tau > 3.05$, in accordance with Theorem 1. An example trajectory of the system is shown in Figure 1(a) under the periodic switching signal

$\sigma(t)$ with a dwell time of $\tau = 1$ (Figure 1(b)). The result shows that the worst case condition (9) is not tight. Furthermore, in this special one-dimensional case the trajectory never leaves the region between the two equilibrium points of the respective subsystems.



(a) (b)

Figure 1: The system trajectory $x(t)$ (a) and the switching signal $\sigma(t)$ (b) of Example 1.

4.2 Example 2

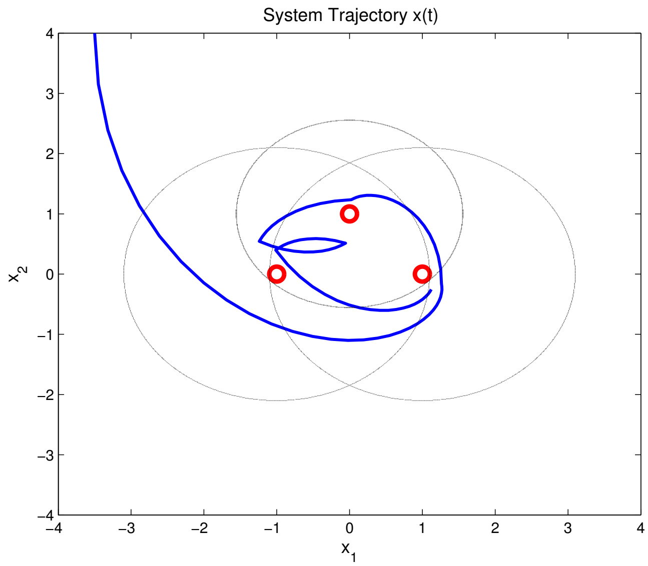
We now choose a 2-dimensional system, where $x := [x_1, x_2]^T$, with three subsystems $Q = \{1, 2, 3\}$ given by

$$\dot{x}^{(i)} = A\, x^{(i)} + b^{(i)},$$

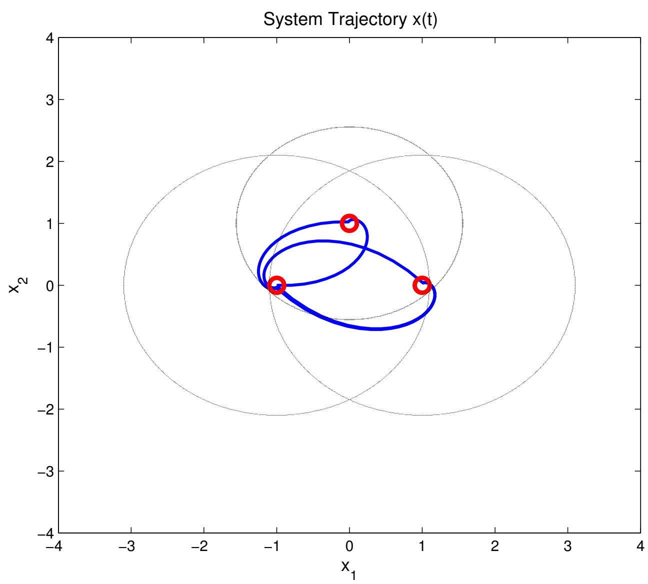
where

$$A = \begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix}, \quad b^{(1)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, b^{(2)} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, b^{(3)} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Let us take the corresponding Lyapunov functions as $V^{(1)}(\mathbf{x}) := x_1^2 + (x_2 - 1)^2$, $V^{(2)}(\mathbf{x}) := (x + 1)^2 + x_2^2$, and $V^{(3)}(\mathbf{x}) := (x_1 - 1)^2 + x_2^2$, respectively. Then, the ε parameter in (3) is $\varepsilon = 2$. If we choose $\kappa = 0.02$, then $\mu = 441$ and the respective dwell time is $\tau > 1.32$. An example trajectory of the system under the random switching signal $\sigma(t)$ with a dwell time of $\tau = 1$ and $\tau = 4$ are shown in Figure 2(a) and Figure 2(b), respectively. Furthermore, Figure 2 shows the set $\mathcal{L}$ (union of the ellipsoid sets bounded by thin lines in the figure) which is calculated numerically in this case.



(a)



(b)

Figure 2: The system trajectory $x(t)$ of Example 2 under the random switching signal $\sigma(t)$ with a dwell time of $\tau = 1$ (a) and $\tau = 4$ (b).

5 Conclusion

We have investigated stability of switched systems with multiple (distinct) equilibria, and proven global convergence to a superset of these equilibrium points. Thus, we have extended available results on the stability of a single equilibrium point in switching systems to multiple equilibrium points. It is noteworthy that the sufficient condition for stability is similar to the one in the single equilibrium case. On the other hand, in the multiple equilibria case, set oriented results replace point stability theorems, leading to interesting observations such as a special type of invariance of the supersets defined.

The results presented are general and can be applied to any switched system formulation. As an example, we have utilized early versions of these results within game theoretical models where the switching is from one game to another, and the unique Nash equilibrium of a particular game corresponds to the equilibrium point of a subsystem [1, 2].

Future research directions include investigation of the case when there are infinitely many subsystems, i.e. an infinite index set Q, and quantifying the relationship between the variable $\mu(\kappa)$ and its argument κ.

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