Loss of Continuity in Cellular Networks Under Stabilizing Transmit Power Control

Vishwesh Kulkarni Vincent Fromion Tansu Alpcan Michael Safonov

***Abstract*— Recently, passivity-based techniques to control the transmit power of mobile nodes have been proposed to ensure the finite gain stability of a class of cellular CDMA networks. These techniques implement the Zames-Falb multipliers at the mobile node and at the base stations. The finite gain stability of such a cellular network follows as a consequence of the fact that the Zames-Falb multipliers preserve positivity of monotone memoryless nonlinearities. In this note, we show that such a cellular network may not be a continuous system. Hence, the following undesirable scenarios may occur if the set-point is varied with time: (i) the Nash equilibrium may cease to exist, and (ii) vanishingly small changes in set-points (such as the target SINR values) may cause the system trajectory to jump from one equilibrium point to another, leading to nonvanishingly small variations in the mobile transmit powers and the actual SINR values across such networks.**

***Index Terms*— CDMA, power control, Nash equilibrium, multipliers**

I. Introduction

Recently, [1] has proposed a passivity-based team-optimized solution to the problem of minimizing transmit power in multi-cell CDMA networks subject to the constraint that the steady state signal-to-noise-and-interference ratio (SINR) exceeds a target value for each mobile node. It establishes a class of Zames-Falb multiplier-based nonlinear dynamic controllers, to be implemented at the base station and the mobile nodes, that ensures the input-output stability of the closed-loop system, provided the instantaneous transmit power of every mobile node stays below a certain threshold for all time instants (see [1, Lemma 2]). [2] has since established that the input-output stability of the closed-loop system of interest holds regardless of this threshold. [3] extends the technique of [1] to implement nonlinear dynamic controllers at the base station. Now, [1] and [3] assume that the target SINR values are constant. In practice, however, such is not the case. Instead, these target SINR values are continually adapted at the base station using an outer-loop power control mechanism. Hence, it is of interest to examine if the power control algorithms proposed by [3] ensure *continuity*, and not just the input-output stability, of the closed-loop system. If the continuity is not ensured, vanishingly small variations in the target SINR values can cause arbitrarily large variations in the actual SINR and transmit power values. In this paper, we show that the transmit power control algorithms derived in [3] may fail to ensure continuity of the closed-loop system, in general.

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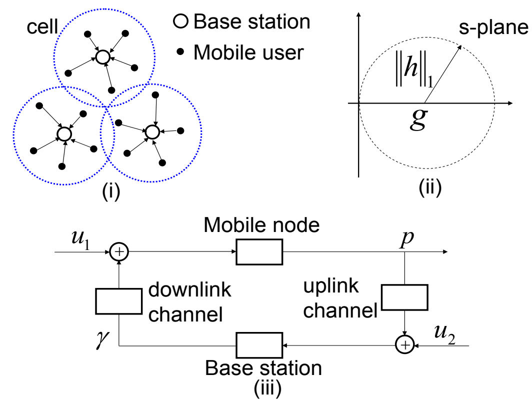

Fig. 1. (i): Cellular network has one base station per cell; a mobile node transmits data to only one base station. (ii) The shaded open circle in the right-half s-plane denotes the feasible region for the Nyquist plot of a Zames-Falb multiplier $m(\cdot) = g\delta(\cdot) + h(\cdot)$. [3] uses the Zames-Falb multipliers to improve the performance of the system shown in (iii).

II. System Description

For simplicity, we shall consider a single cell network. The system description is as follows (see Fig. 1). The base station receives data from N mobile nodes over time-varying wireless channels. The uplink transmit power for the i-th mobile node is $p_i \in [0, p_{max}]$, where p_{max} is a pre-set upper bound on the transmit powers. The corresponding signal received at the base station is $x_i = h_i p_i$ where $h_i \in [0, 1]$ is the channel gain. Thus, the SINR of mobile i at the base station is

$$\gamma_i = \frac{L h_i p_i}{\sum_{j, j \neq i} h_j p_j + \sigma^2},$$

where $L > 1$ is the spreading gain of the network and $\sigma^2 > 0$ is the variance of the background noise. Let us set the cost function $C_i(p_i)$ of the i-th mobile user to be

$$C_i(p_i) = J_i(p_i) - U_i(\gamma_i),$$

where the *energy function* J_i is a twice continuous differentiable, nondecreasing, convex function capturing the energy cost incurred by the user in transmitting its data, and the *utility function* U_i, defined by $U_i(\gamma_i) \doteq \log(\gamma_i + L)$, is a concave function, motivated by the maximal achievable bandwidth given by Shannon's theorem (see [3]), capturing the willingness of the user to adjust its SINR γ_i. The utility function may be scaled by a user specific scalar paramater for added flexibility. However, for simplicity, we will assume that the scaling factor is unity. The optimization problem can now be stated as follows:

$$\min_p \sum_{i=1}^{N} C_i(p_i) \quad \text{s. t.} \quad \gamma_i \geq \overline{\gamma}_i, 0 \leq p_i \leq p_{max} \quad \forall i, \tag{1}$$

where $p \doteq [p_1 \ p_2 \ \dots \ p_N]^T$, $\overline{\gamma}_i$ is the target SINR for the mobile i, and p_{max} is chosen sufficiently high so that $p_i(t) \leq p_{max} \ \forall t$. Let

$$\begin{aligned} A &\doteq \begin{bmatrix} h_1 & -h_2\frac{\overline{\gamma}_1}{L} & \dots & -h_N\frac{\overline{\gamma}_1}{L} \\ -h_1\frac{\overline{\gamma}_2}{L} & h_2 & \dots & -h_N\frac{\overline{\gamma}_2}{L} \\ \vdots & \vdots & \dots & \vdots \\ -h_1\frac{\overline{\gamma}_N}{L} & -h_2\frac{\overline{\gamma}_N}{L} & \dots & h_N \end{bmatrix} \\ b &\doteq [\overline{\gamma}_1 \frac{\sigma^2}{L} \ \dots \ \overline{\gamma}_N \frac{\sigma^2}{L}]^T \\ \Omega &\doteq \{p \in \mathbb{R}^N : Ap \geq b, \ p_i \in [0, p_{max}] \ \forall i\}. \end{aligned}$$

Then, the optimization problem given by (1) is recast as

$$\min_p \sum_{i=1}^{N} C_i(p_i) \quad \text{subject to} \quad p \in \Omega. \tag{2}$$

III. Background Results

Since the problem is strictly convex, if Ω is not empty, there exists a unique solution. With respect to the feasibility of this convex optimization problem, [1] has proved the following result on the existence and uniqueness of the solution.

Lemma 1: (Lemma 3.1 of [1])
If Ω is nonempty, the optimization problem given by (1) has a unique global minimum. □

Lemma 2: (Lemma 3.2 of [1])
If $\theta \doteq \sum_j \overline{\gamma}_j/(\overline{\gamma}_j + L) < 1$ and if p_{max} is chosen sufficiently large, then Ω is nonempty and every p satisfying $Ap \geq b$ satisfies $p > 0$. Furthermore, Ω is empty if $\theta \geq 1$. □

If $\theta < 1$ and if p_{max} is large enough, Lemma 2 establishes an optimal solution p^* to the optimization problem given by (2). It may be noted that p^* minimizes the Lagrangian

$$L(p, \lambda) \doteq \sum_i C_i(p_i) - \lambda^T(Ap - b). \tag{3}$$

Since A is full-rank, λ is unique. Let $q \doteq A^T\lambda, r \doteq \text{diag}(p_i)q$. Then, (3) is recast as

$$L(p, \lambda) \doteq \sum_i (C_i(p_i) - r_i) - \lambda^T b,$$

with p^* satisfying $\frac{dC(p)}{dp}|_{p=p^*} - q = 0$. Let us define the convex user and network problems as follows:

$$\begin{aligned} \text{user } i\text{:} \ &\min_{r_i} C_i(r_i/q_i) - r_i, \ r_i \geq 0 \\ \text{network:} \ &\min_p \sum_i -r_i \log(p_i), \quad p \in \Omega. \end{aligned}$$

[1] shows that there exist p, q, r with $r_i = p_i q_i$ such that r_i solves the user i problem and p solves the network problem so that p is the unique solution to the optimization problem given by (1). Then, [1] implements the decentralized nonlinear optimal control through the following primal update algorithm:

$$\begin{aligned} &\underline{\text{Base Station Price Update:}} \\ &q = A^T f(Ap), \\ &\underline{\text{Mobile User Power Update:}} \\ &\Sigma_i : \ \dot{p_i} = -k_i \left(\frac{dC_i}{dp_i} - q_i\right) \quad \text{with } k_i > 0, \end{aligned}$$

where $f(Ap) \doteq [\psi_1(\text{row}_1(Ap)) \dots \ \psi_K(\text{row}_K(Ap))]^T$, and the user-specific $\psi_i(\zeta)$ is memoryless monotone continuous and zero-valued if its argument is negative-valued. Note that the controller implemented at the base station to generate the feedback q is static. [3] goes one step beyond and implements a dynamic controller at the base station to generate the feedback q. [3] implements the decentralized nonlinear optimal control through the following primal update algorithm:

$$\begin{aligned} &\underline{\text{Base Station Price Update:}} \\ &q = MA^T f(Ap), \\ &\underline{\text{Mobile User Power Update:}} \\ &\Sigma_i : \ \dot{p_i} = -k_i \left(\frac{dC_i}{dp_i} - q_i\right) \quad \text{with } k_i > 0, \end{aligned}$$

where M is a Zames-Falb multiplier that we will formally define shortly, and other functions and variables are as defined earlier. Let us refer to this closed-loop feedback system as $\mathcal{S}_1$. To facilitate the input-output stability analysis of $\mathcal{S}_1$, we now note down relevant notions and results.

Definition 1: The *causal truncation* $P_T f$ of a function f is defined as

$$P_T f(t) = \begin{cases} f(t) & \text{if } t \leq T; \\ 0 & \text{else,} \end{cases}$$

where $T \in [0, \infty)$. □

Definition 2: The *extended space* $\mathcal{L}_{2e}$ comprises functions whose causal truncations are in $\mathcal{L}_2$. Given $f \in \mathcal{L}_{2e}$ and $T \in [0, \infty)$, we write $\|P_T f\|$ as simply $\|f\|_T$. □

Definition 3: A system $\mathcal{S}$ mapping $u \in \mathcal{L}_2$ into $y \in \mathcal{L}_2$ is said to be *finite gain stable* if there exists $\gamma \geq 0$ such that$\|\mathcal{S}(u)\| \leq \gamma\|u\|$ for all $u \in \mathcal{L}_2$. □

Definition 4: A system $\mathcal{S}$ mapping $u \in \mathcal{L}_2$ into $y \in \mathcal{L}_2$ has a finite *incremental gain* if there exists $\eta \geq 0$ such that $\|\mathcal{S}(u_1) - \mathcal{S}(u_2)\| \leq \eta\|u_1 - u_2\|$ for all $u_1, u_2 \in \mathcal{L}_2$. □

TABLE I
NOTATION

Symbol	Meaning
$(\mathbb{R}^+)$ $\mathbb{R}$	Set of all (nonnegative) real numbers.
$\mathbb{Z}$	Set of all integers.
$(\cdot)'$ or $(\cdot)^T$	Transpose of a vector or a matrix $(\cdot)$.
$\langle x, y\rangle$	$= \int_{-\infty}^{\infty} y^T(t)x(t)dt$
$\|x\|$	$= \sqrt{\langle x, x\rangle}$.
$\mathcal{L}_2$	Space of possibly vector valued signals x for which $\|x\| < \infty$.
$\|z\|_1$	$= \int_{-\infty}^{\infty} \|z(t)\|\, dt$.
$x(t) * y(t)$	$= \int_{-\infty}^{\infty} x(\tau)y(t-\tau)d\tau$
x^*	$x^*(t) = x^T(-t)$ if $x(t) \in \mathbb{R}^n$.
$\widehat{x}$	$= \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\, dt$
$r_{xy}(t)$	$= x * y^* = \int_{-\infty}^{\infty} x(t+\tau)y^T(\tau)d\tau$
$\delta(t)$	$= \begin{cases} 1 & \text{if } t = 0, \\ 0 & \text{else.} \end{cases}$
$\mathrm{diag}(p_i)$	Diagonal matrix with p_i as its diagonal elements.

Definition 5: A system $\mathcal{S}$ mapping $u \in \mathcal{L}_2$ into $y \in \mathcal{L}_2$ is said to be *incrementally stable (Lipschitz continuous)* if it is finite gain stable with a finite incremental gain. □

Definition 6: A system $\mathcal{S}$ mapping $u \in \mathcal{L}_2$ into $y \in \mathcal{L}_2$ is said to be continuous at $u_1 \in \mathcal{L}_2$ if it maps $\mathcal{L}_2$ into $\mathcal{L}_2$ and, in addition, if for any $\epsilon > 0$ there exists $\delta > 0$ such that $\|\mathcal{S}(u_1) - \mathcal{S}(u_2)\| \le \epsilon$ for any $u_2 \in \mathcal{L}_2$ such that $\|u_1 - u_2\| \le \delta$. □

Definition 7: Given $a, b \in \mathbb{R}$ with $b > a$, a nonlinearity $N : \mathbb{R}^n \mapsto \mathbb{R}^n$ is said to be *in sector*$[a, b]$ if $\langle N(x) - ax, N(x) - bx\rangle \le 0\ \forall x \in \mathbb{R}^n$. □

Definition 8: Given $a, b \in \mathbb{R}$ with $b > a$, a nonlinearity $N : \mathbb{R}^n \mapsto \mathbb{R}^n$ is said to be *incrementally in sector*$[a, b]$ if $\langle N(x) - N(y) - a(x-y), N(x) - N(y) - b(x-y)\rangle \le 0\ \forall x, y \in \mathbb{R}^n$. □

Definition 9: The class $\mathcal{N}_M$ of *monotone nonlinearities* consists of all memoryless mappings $N : \mathbb{R}^n \mapsto \mathbb{R}^n$ such that:

1) N is the gradient of a convex real-valued function; and
2) there exists $C \in \mathbb{R}^+$ s.t. $\|N(x)\| \le C\|x\| \quad \forall x \in \mathcal{L}_2$.

The class $\mathcal{N} \doteq \{N \in \mathcal{N}_M | N(0) = 0\}$ and the class $\mathcal{N}_{odd} \doteq \{N \in \mathcal{N} | N(x) = -N(-x)\ \forall x\}$. □

Definition 10: `[Zames-Falb multipliers]`
The class $\mathcal{M}_{ZF}$ of Zames-Falb multipliers denotes the class of convolution operators, either continuous-time or discrete-time, such that the impulse response of an $M \in \mathcal{M}_{ZF}$ is of the form

$$m(\cdot) = g\ \delta(\cdot) + h(\cdot) \quad \text{with} \quad \|h\|_1 < g,\ h(t) \le 0\ \forall t$$

where $g, h(\cdot) \in \mathbb{R}$. The class $\mathcal{M}_{ZF}^{odd}$ is obtained if the condition $h(t) \le 0 \forall\ t$ is relaxed. □

Remark 1: A multiplier preserves positivity of a $\mathcal{N}_M$ ($\mathcal{N}_M^{odd}$) nonlinearity if and only if it is in $\mathcal{M}_{ZF}$ ($\mathcal{M}_{ZF}^{odd}$) (see [4] and [5]). □

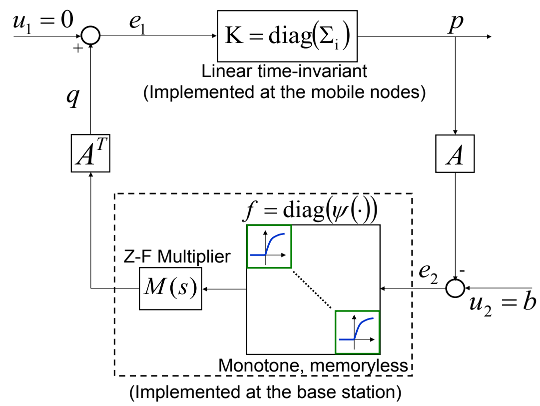

Fig. 2. Block-diagram decomposition of $\mathcal{S}_2$. [3] sets the exogenous inputs u_1 and u_2 to 0 and b, respectively, where the choice of b is as per the base station administrator. The feedback nonlinearity $f(\cdot)$ is repeated monotone single-input single-output (SISO). [3] synthesizes a dynamic feedback nonlinearity at the base station by multiplying the output of $f(\cdot)$ by a Zames-Falb multiplier $M(s)$.

The Zames-Falb multipliers reduce conservatism in passivity-based finite gain stability analysis as follows (see [6]).

Theorem 1: `[finite gain stability theorem]`
Consider a feedback system $\mathcal{S}_f$ described by the following equations:

$$y_1 = He_1, \quad e_1 = u_1 - N(e_2), \quad e_2 = y_1 + u_2, \tag{4}$$

where H is a linear time-invariant system, $N \in \mathcal{N}$ ($N \in \mathcal{N}$) and u_1, u_2 are constant valued signals. Then, $\mathcal{S}_f$ is finite gain stable if it holds that $Re(\widehat{M}(j\omega)\widehat{H}(j\omega)) > 0 \quad \forall \omega \in \mathbb{R}$ for at least one $M \in \mathcal{M}_{ZF}$ ($M \in \mathcal{M}_{ZF}^{odd}$). If, in addition, $N \in \mathrm{sector}[a, b]$, then $\mathcal{S}_f$ is finite gain stable if it holds that

$$Re\left(\widehat{M}(j\omega)\frac{\widehat{H}(j\omega) + 1/b}{1 + a\widehat{H}(j\omega)}\right) > 0 \quad \forall \omega \in \mathbb{R},$$

for at least one $M \in \mathcal{M}_{ZF}$ ($M \in \mathcal{M}_{ZF}^{odd}$). □

Let $\mathcal{S}_2$ denote the system obtained from $\mathcal{S}_1$ by inserting a Zames-Falb multiplier at the output of $f(\cdot)$. Then, the following result is readily established using the passivity theorem and Remark 1 (see [1]).

Theorem 2: Consider $\mathcal{S}_2$ with $M \in \mathcal{M}_{ZF}$. Then, $\mathcal{S}_2$ is finite gain stable at the Nash equilibrium p^*. □

Further, using classical results between the $\mathcal{L}_2$ gain and Lyapunov stability, it follows that the equilibrium states associated with ($u_1 = u_2 = 0$) are globally exponentially stable. In practice, as of now, u_2 is time-varying because the SINR target values are time-varying in the current IS-95 CDMA implementations. Specifically, the closed-loop power control system implemented in IS-95 CDMA comprises an outer loop power control algorithm that updates the SINR thresholds every 10 ms and an inner loop power control algorithm that updates the transmit power at 800 Hz (see [7]); a block diagram representation of this control system is

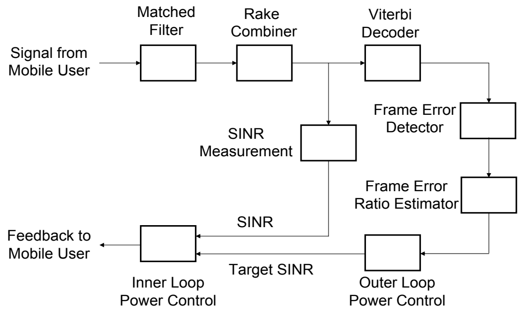

Fig. 3. Block diagram of the inner-outer loop power control mechanism implemented in CDMA system (see [7]). Power control algorithm obtained as a solution to a Nash game problem, as in [1], could be implemented in the inner loop power control module shown in this figure. We show that such a closed loop system might not have a unique Nash equilibrium, if at all, if the target SINR is not constant.

shown in Fig. 3. We are interested in understanding how the time-varying u_2 affects the stability, and even the existence, of the Nash equlibrium. Now, as the first step, a direct consequence of Theorem 1 gives us the following result.

Lemma 3: Consider $\mathcal{S}_2$ with $M \in \mathcal{M}_{ZF}$. Suppose u_1 and u_2 are constant inputs. Then $\mathcal{S}_2$ possesses an unique equilibrium regime which is globally exponentially stable if the state-space representation of $H(s)$ is minimal. □

We will now demonstrate that $\mathcal{S}_2$ ceases to possess this nice property if u_2 is time-varying and *not* constant. In fact, it turns out that $\mathcal{S}_2$ might have several steady-state equilibrium regimes as a function of u_2, the reason being that $\mathcal{S}_2$ may not be uniformly continuous. In fact, we will show that the uniform continuity, or, the stronger notion of incremental stability, of the closed-loop system implies the unique steady state property. Theorem 2 rests on the fact that a Zames-Falb multiplier preserves positivity of *all* positive memoryless monotone nonlinearities. However, a Zames-Falb multipliers does not, in general, preserve the incremental positivity of all incrementally positive memoryless monotone nonlinearities. As a result, these multipliers may reduce conservatism in the multiplier-based incremental stability analysis conditions only if the multipliers are constant (see [8]). Hence, it is possible that vanishingly small variations in the set points (such as b, p_{min}, and p_{max}) may induce arbitrarily large variations in the mobile transmit powers if non-static Zames-Falb multipliers are used for transmit power control in $\mathcal{S}_2$. We now demonstrate that such is indeed the case.

IV. Main Result

Consider the cellular network with only one mobile node. Let the target SINR value of the node be 10 db. Let the channel gain $h_1 = 1$ and let the mobile user power update algorithm be given by

$$\Sigma_1 = \frac{1}{s+100}.$$

Note that the transfer function Σ_1 used to generate the mobile transmit power p_1 as a function of the base station feedback signal is a low pass filter with the bandwidth of approximately 314 Hz. Now, suppose the base station implements the heavily underdamped transfer function $M(s) = 909(s^2+10.1s+1)/(s^2+0.1s+1)$ as the class $M \in \mathcal{M}_{ZF}^{odd}$ Zames-Falb multiplier, and uses the following odd nonlinearity f in its price update algorithm:

$$f(x) = \begin{cases} -1 & \text{if} \quad x \le -1-\delta; \\ ax^3+bx^2+cx+d & \text{if} \quad x \in [-1-\delta, -1+\delta]; \\ x & \text{if} \quad x \in [-1+\delta, 1-\delta]; \\ ax^3-bx^2+cx-d & \text{if} \quad x \in [1-\delta, 1+\delta]; \\ 1 & \text{if} \quad x \ge 1+\delta, \end{cases}$$

where $\delta = 0.01$, $a = -2.47\ 10^{-6}$, $b = -0.1244$, $c = 0.2512$, and $d = -0.6194$. Thus the nonlinear dynamic price update algorithm implemented by the base station is given by $q = Mf(p_1)$. Suppose the background noise is given by $\sigma^2 = 0.05$ and the spreading gain L is unity. Thus, $u_2 = b = 0.5$. Let us refer to this system as $\mathcal{S}_0$. Note that, since $0 \le \frac{\partial f(x)}{\partial x} \le 1$, f is incrementally inside sector $[0,1]$. Thus, $\mathcal{S}_0$ is an instance of $\mathcal{S}_f$ with $N = f$ and

$$H(s) = \frac{909(s^2+10.1s+1)}{(s^2+0.1s+1)(s+100)}.$$

Note that

$$\inf_{w\in\mathbb{R}} Re\left(\widehat{H}(j\omega)+1\right) > 0.$$

Hence, by Theorem 2, it follows that $\mathcal{S}_0$ is finite-gain stable. We now show that it is not incrementally stable. To do so, we first note that a causal operator $\mathcal{S} : \mathcal{L}_{2e} \to \mathcal{L}_{2e}$ has a linearization, viz., a Gâteaux derivative, at $u_0 \in \mathcal{L}_{2e}$ if for any $T \in [t_0, \infty)$ and for any $h \in \mathcal{L}_{2e}$, there exists a continuous linear operator $D\mathcal{S}_G[u_0] : \mathcal{L}_{2e} \to \mathcal{L}_{2e}$ such that

$$\lim_{\lambda\to 0}\left\|\frac{\mathcal{S}(u_0+\lambda h)-\mathcal{S}(u_0)}{\lambda} - D\mathcal{S}_G[u_0](h)\right\|_T = 0.$$

In addition, it is well known that if $\mathcal{S}$ has a linearization for any input in $\mathcal{L}_{2e}$, then $\mathcal{S}$ is incrementally stable only if all its minimal state-space linearizations are exponentially stable. We now state a result that will help us prove that $\mathcal{S}_0$ is not incrementally stable.

Lemma 4: Consider the system $\mathcal{S}_f$ described by (4). Let (A,B,C) be the state-space realization of $H(s)$. Suppose f is incrementally in sector $[a,b]$. Then, the system $\mathcal{S}_f$ is not incrementally stable if there exists a time-varying and measurable matrix $A(t) \in \mathcal{A} \doteq \{A(t) \doteq A + Bk(t)C \mid k(t) \in [a,b]\}$ such the following associated linear system, $\dot{z}(t) = A(t)z(t)$, is not finite gain stable. □

Proof: The linearization of $\mathcal{S}_f$ along a specific input $u_r = [u_{1r}u_{2r}]^T$ corresponds to an interconnection between a linear time-invariant system $H(s)$ that has a state-space representation given by

$$\dot{\xi} = A\xi + Be_{1r} \quad \text{with} \quad \xi(t_0) = 0, \quad y_{1r} = C\xi,$$

and the time-varying gain $e_{1r} = u_{1r} - k(y_{1r} + u_{2r})$, where

$$k \doteq \frac{\partial f(y_{1r} + u_{2r})}{\partial \xi},$$

and y_{1r} is associated to $\mathcal{S}_f$ for the input $u_r = [u_{1r}\ u_{2r}]^T$ and for the initial condition ξ_0. The realization of the linearization is bounded and minimal since the state-space representation of $H(s)$ is minimal and, further, $k(t)$ is bounded (see [9, Lemma 3]). Hence, a necessary condition for the incremental stability of Σ is the exponential stability of linearizations defined by

$$\dot{z} = Az - BkCz \tag{5}$$

where, by definition, $k(t) \in [a, b]\ \forall\ t \in [t_0, \infty)$. In the other hand, $\mathcal{A}$ defined a linear differential inclusion given by

$$\dot{z}(t) = Az(t) - Bw(t)Cz(t) \tag{6}$$

where input $w(t)$ is a measurable signal such that

$$a \le w(t) \le b \quad \forall\ t \in [t_0, \infty).$$

Note that if the linear differential inclusion given by (6) is not finite gain stable, then a linearization of $\mathcal{S}_0$ is not finite gain stable. Thus, the necessity is proved if for the same initial condition, the solutions of system (6) are the solutions of system (5), *i.e.*, for any measurable input $w(t)$ such that $a \le w(t) \le b$, there at least exists an input u_{2r} belonging to $\mathcal{L}_{2e}$, such that

$$w = \frac{\partial f}{\partial \xi}(y_{1r} + u_{2r}) \text{ a.e.} \tag{7}$$

This fact can be proved as follows. We first establish that it is possible to choose the input of f as follows. Suppose the input of f is ν. Let us consider the output of the open-loop system associated to the connection between $H(s)$ and f, i.e., $y_{1\nu} = H(f(\nu))$. Now, consider the closed-loop system and define $u_{2r} \doteq \nu - y_{1\nu}$. Then, by definition, the input of f is ν. We next prove that for any measurable input w, there exists $u_{2r} \in \mathcal{L}_{2e}$ such that (7) is satisfied. To do so, let us recall that a measurable function is the limiting case of a step function (see [10]). Hence, for any given measurable w, there exist a sequence of step functions, ϕ_n such that

$$\lim_{n\to\infty} \phi_n = u \text{ a.e.}$$

Moreover, since $\frac{\partial f}{\partial x}$ is a continuous function, for any ϕ_n, there exists a step function ψ_n, such that

$$\frac{\partial f}{\partial \sigma}(\psi_n) = \phi_n.$$

Hence, there exists an input belonging to $\mathcal{L}_{2e}$ defined by

$$u_{2r} = \lim_{n\to\infty} \psi_n - y_{1r}$$

and such that (7) is satisfied. Indeed, u_{2r} is the sum of two functions belonging to $\mathcal{L}_{2e}$ since the closed-loop is assumed well-defined that ensures that $e_{ir} \in \mathcal{L}_{2e}$ and $|\psi_n|$ is bounded by $K \doteq \max\{|a|, |b|\}$. We thus deduce that g defined by $g(t) \doteq \lim_{n\to\infty} \psi_n(t)$, is square integrable on any finite support since $\|g(t)\|$ is a bounded and measurable function on any finite interval of time. Hence the proof. □

Following Lemma 4, we select the following $\pi/2$-periodic matrix in $\mathcal{A}$:

$$A(t) = \begin{cases} A - BC & \text{if } t \in [0, 0.34]; \\ A & \text{if } t \in (0.34, \pi/2], \end{cases}$$

where

$$A = \begin{bmatrix} -100.1 & -0.0859 & -0.0061 \\ 128 & 0 & 0 \\ 0 & 128 & 0 \end{bmatrix},$$
$$A - BC = \begin{bmatrix} -100.1 & -0.0859 & -0.0616 \\ 128 & 0 & 0 \\ 0 & 128 & 0 \end{bmatrix}.$$

The state transition matrix associated with the system $\mathcal{S}_v$ defined by $\dot{z}(t) = A(t)z(t)$ is

$$\Phi(T_0, 0) = e^{(A-BC)\tau} e^{A(\pi/2-\tau)} = \begin{bmatrix} -0.05 & -0.04 & 0 \\ -2.9 & -2.3 & -0.01 \\ 84.8 & 66.1 & -0.06 \end{bmatrix},$$

where $\tau = 0.34$. The eigenvalues of $\Phi(T_0, 0)$ have absolute values $|\lambda_1| = 1.9847$, $|\lambda_2| = 0$, $|\lambda_3| = 0.4441$. Since $|\lambda_1| > 1$, a routine application of the Floquet theorem shows that the periodic system $\mathcal{S}_v$ is not exponentially stable and, hence, not finite gain stable. Therefore, by Lemma 4.1, $\mathcal{S}_0$ is not incrementally stable.

Remark 2: The nonlinear closed-loop system $\mathcal{S}_v$ has p^* as the steady-state value for all initial conditions so long as the forcing inputs, namely the target SINRs, are held constant for all time instants. Note that different forcing inputs may lead to different steady-state values. Our main result shows that if the forcing inputs are not held constant, then even the existence of a unique steady-state p^* is not guaranteed under the Zames-Falb multiplier based approach. □

Remark 3: Given a nonlinear system satisfying the conditions of Lemma 4, all the constant matrices belonging to the polytope $\mathcal{A}$ are necessarily stable (see [11]). Hence, our counterexample uses a *time-varying* $A(t)$. Our counterexample, given by $\mathcal{S}_0$, rests on the jump resonance exhibited by a system that satisfies the Popov criterion. Describing function analysis allows to one to deduce the frequency and the amplitude of a periodic input around which the associated time-varying periodic linearization is possibly unstable (see [12, Ch. 5] and [13]). Indeed, a describing function analysis of $\mathcal{S}_0$ predicts a jump resonance for the following periodic square-wave input u_2 of period 2τ that switches between the values 0 and 1. This input arises in $\mathcal{S}_0$ if the target SINR is varied around its baseline value of 10db by a periodic square-wave having the amplitude of 5db and the period of 2τ. Now, the system has three possible steady-states for the same u_2: two of these steady-states are stable whereas the other one is unstable (see Fig. 4). □

Remark 4: Our counterexample relies on using a heavily underdamped filter that has a very small bandwidth (approximately 3.14 Hz), compared to the bandwidth of the mobile

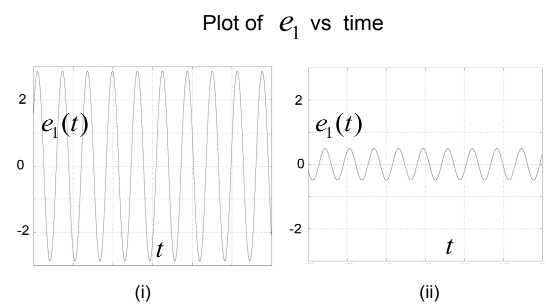

Fig. 4. For the system considered in our counterexample, the error signal e_1, specified in Figure 2, is given by one of the two limit cycles shown in (i) and (ii) if the target SINR is varied periodically between 5db and 15db at a frequency of 6 Hz. In this case, u_2 turns out to be a square wave of amplitude 0.5 and period 0.16 seconds with a DC offset of 0.5. In our counterexample, the target SINR is updated at a rate of approximately 6 Hz, much lower than the 100 Hz update rate used in the current IS-95 CDMA standard.

user power update controller, as the Zames-Falb multiplier at the base station. It is not clear whether the desired continuity can be obtained by restricting the Zames-Falb multipliers in the base station price update algorithms to be overdamped filters that have a sufficiently large bandwidth. □

V. Discussion

The problem of determining whether a given system is incrementally stable or not is equivalent to solving a set of Hamilton-Jacobi-Isaac inequalities (see [14] and [15]) or, alternatively, determining the stability of a linear differential inclusion (see [16]). If the given system is a Lure'-type system, this problem is NP-hard. Sufficient conditions for incremental stability were established in [17] using incremental conicity properties (also see [18]). However, as [16] shows, these conditions can be quite conservative prompting the question as to whether multipliers used to establish the finite gain stability of Lure' systems can be used to reduce this conservatism. Unfortunately, the answer is not affirmative as [8] shows that the Zames-Falb multipliers and Popov multipliers do not preserve incremental positivity of monotone nonlinearities. Our counterexample in Section IV illustrates its consequence in CDMA power control.

So, what is the worst that can happen in CDMA power control under a continuous update of the target SINR? If the target SINR is a periodic signal of frequency ω, then the describing function analysis (see [12]) may be used to attempt a characterization of the possible jumps in the closed-loop system as follows. Let us assume that the solution of the closed-loop system is a periodic signal. Then, we can replace the nonlinearity at the base station by its describing function using the first harmonic approximation. Now, it remains to solve the closed-loop system which is system of nonlinear equations that depend not only on ω but also on the amplitude of the first harmonic of the nonlinearity (see [13], [19], and [20]). Under some realistic assumptions, this method leads to rigorous analysis (see [21] and [22]) if the linear-time invariant element in the closed-loop system is a low pass filter. Techniques to check for the existence of the jumps are described in [13] and [23, Ch. 12].

If the jumps in the closed-loop system occur slowly enough, a time-scale separation argument on the lines of [24] may be expected to yield a reasonable solution to the CDMA inner-outer loop power control problem. Alternatively, the base station could disable the outer loop power control, set the worst case SINR targets and allow the mobile users to go beyond these in a best effort manner.

VI. Conclusion

Prevalent CDMA power control algorithms employ a so-called outer loop power control algorithm at a base station to update the target SINR values for the mobile users served by the base station. An inner loop power control algorithm is implemented at the mobile user end to update its transmit power as a function of a feedback signal received from the base station. Recently, [1] and [3] have posed the CDMA power control problem as a Nash game and has established a class of Zames-Falb multiplier-based nonlinear dynamic controllers, to be implemented at the base station and the mobile nodes, that ensures the input-output stability of the closed-loop system. The solution rests on the existence of a unique Nash equilibrium for the closed loop system. We have shown that the closed loop CDMA system employing the solution proposed by [3] as its inner loop power control may not be a continuous system. As a result, vanishingly small changes in the target SINR values may lead to non-vanishingly small changes in the mobile transmit power and, further, may threaten the existence of the Nash equilibrium.

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