A Control Theoretic Approach to Noncooperative Game Design

Tansu Alpcan, Lacra Pavel and Nem Stefanovic

Abstract— **This paper investigates design of noncooperative games from a control theoretic perspective. Pricing mechanisms are used as a design tool to ensure that the Nash equilibrium of a broad class of noncooperative games satisfies certain global objectives such as welfare maximization. The class of games considered provide a theoretical basis for a variety of decentralized resource allocation and control problems including network congestion control, wireless uplink power control, and optical power control.**

The game design problem is analyzed under full and limited information assumptions for dynamic systems and non-separable utility functions. Stability properties of the game and pricing dynamics are studied under the assumption of time-scale separation and in two separate time-scales. Thus, sufficient conditions are derived, which allow the designer to place the Nash equilibrium solution or to guide the system trajectory to a desired region or point. The obtained results are illustrated with examples.

I. Introduction

Game theory has been used extensively as a quantitative framework for studying communication networks and distributed control systems among its other applications in engineering and economics. Game theoretic models provide not only a basis for analysis but also for design of network protocols and decentralized control schemes [1], [2]. While (noncooperative) game theory has recently enjoyed widespread use in engineering, there is surprisingly little work on *how to design games* such that their outcome satisfies certain global objectives.

While there is a general agreement on the usefulness of game theory, the issues of **price of anarchy** or *efficiency loss* associated with noncooperative games even under the existence of pricing mechanisms, have been the subject of many investigations [3]–[5]. Consequently, a variety of pricing schemes have been proposed in the literature aiming to improve Nash equilibrium (NE) efficiency with respect to a chosen criterion in specific settings [2], [6]–[9]. The research community has revisited the issue of game design again very recently [10], [11]. On the other hand, these studies are limited either to special problem formulations or adopt specific efficiency criteria such as the "system problem" of [12]. A related early line of work focuses specifically on three agent (player) dynamic noncooperative games with multi-levels of hierarchy [13], [14], where it has been shown that the leader has an optimal incentive policy which is linear in the partial dynamic measurement and induces the desired behavior on the two followers. In addition, a separate but substantial literature exists under the umbrella of implementation theory, especially in the field of economics, which focuses on finding fundamental bounds for games where the outcome satisfies some given criteria [15]. These works, however, are not algorithmic and do not have an engineering perspective.

The **main contribution** of this paper is a treatment of **game design from a control theoretic perspective**. A fairly general class of games are considered, which have been applied to a variety of settings including network congestion control, wireless uplink power control, and optical power control [16]–[20]. Extending the results of [21], the focus here is on analysis of dynamic systems arising from game formulations and the general case of non-separable player utilities. In such cases of coupled utility functions, the utility of each player is affected by decisions of other players. In addition, the control theoretic approach adopted here based on game dynamics provides a more realistic model for a variety of applications compared to a static optimization one. Stability properties of game and pricing dynamics are investigated under the assumption of time-scale separation and in two separate time-scales using Lyapunov theory and singular perturbation approach. The sufficient conditions derived, which ensure system stability, are illustrated with examples.

Another dimension of the design problem is the **amount of knowledge** available to the designer. It is rather straightforward to define and achieve a global objective when designing a game under full information. However, the problem gets complicated when the designer operates under limited information. As a starting point, the treatment in this paper is restricted to a class of games where players do not manipulate the game by deceiving the system designer. Such players are sometimes called "*price-taking*" as opposed to manipulative or "price-anticipating". Thus, the utility functions are assumed here to accurately reflect player preferences.

The rest of the paper is organized as follows. Section II presents the underlying model for noncooperative game design. In Section III, a control theoretic formulation for design of games is introduced. Section IV discusses game design under incomplete information, with illustrative examples and numerical simulations in Section V. The paper concludes with remarks of Section VI.

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II. Game Model

The model considered in this paper is similar to the one in [21] and consists of a class of N player static **noncooperative games** on the compact action (strategy) space $\Omega \subset \mathbb{R}^N$ where players' actions are denoted by the vector $\mathbf{x} \in \Omega$, with x_i denoting the i^{th} player's actions. Furthermore, the i^{th} player is associated with a smooth (continuously differentiable) cost function, $J_i : \mathbb{R} \times \Omega \rightarrow \mathbb{R}$, $J_i(\alpha_i, \mathbf{x})$, $i = 1, 2, \ldots, N$, parameterized by a scalar "pricing" or game parameter $\alpha_i \in \mathbb{R}$. In some formulations, there may be (coupled) restrictions on the domain of these parameters such that $\alpha \in \tilde{\Omega} \subset \mathbb{R}^N$. However, it will be assumed that $\tilde{\Omega} = \mathbb{R}^N$ for simplicity unless otherwise stated. Assuming a set of sufficient conditions for the existence of at least one Nash equilibrium (NE) are satisfied, define a game mapping, $\mathcal{T}$ (an inverse game mapping $\hat{\mathcal{T}}$) that maps game parameters α to NE points (NE points to parameters):

$$\mathcal{T} : \mathbb{R}^N \rightarrow \Omega \quad \text{and} \quad \hat{\mathcal{T}} : \Omega \rightarrow \mathbb{R}^N, \tag{1}$$

such that

$$\mathbf{x}^* = \mathcal{T}(\alpha^*) \quad \text{and} \quad \alpha^* = \hat{\mathcal{T}}(\mathbf{x}^*) \tag{2}$$

for any NE point $\mathbf{x}^*$ and corresponding parameter vector α^*. Notice that the mappings $\mathcal{T}$ and $\hat{\mathcal{T}}$ are highly nonlinear, often not explicitly expressible, and may not be one-to-one or invertible except for special cases, i.e. games with special properties.

Next, consider **a specific class of games**, $\mathcal{G}1$, by assuming a cost structure of the form

$$J_i(\alpha_i, \mathbf{x}) = \alpha_i p_i(\mathbf{x}) - U_i(\mathbf{x}), \tag{3}$$

where the functions p_i and U_i are smooth and chosen in such a way that there exists at least a single NE in the game, e.g. the function p_i can be convex while U_i is strictly concave with respect to x_i for any given $\mathbf{x}_{-i}$. Further define **another class of games**, $\mathcal{G}2$, as a special case of $\mathcal{G}1$ with additional conditions on the cost structure, such that they admit a unique NE solution. An extensive analysis on conditions for the existence and uniqueness of NE can be found in [1]. Notice that a large set of network games belong to this class with notable examples of network congestion games [2], [22], power control games in wireless networks [16] and optical networks [19].

Assume that the utility function U_i accurately reflects the user preferences. Then, the pricing function p_i and parameters α enable the system designer to **control the game dynamics** and outcome to achieve a desired objective. This is similar -in spirit- to the goal of implementation theory or mechanism design in the economics literature [15] with the important difference of not allowing users to knowingly manipulate the system, i.e. assuming price-taking users only. This game design problem is illustrated in Figure 1.

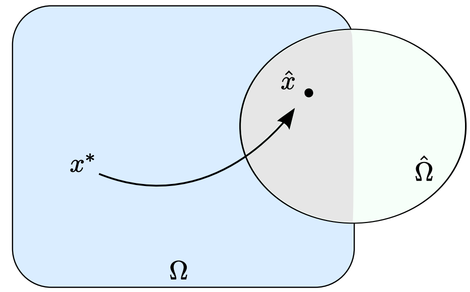

Fig. 1. Game design involves controlling the game dynamics such that the NE, $\mathbf{x}^* \in \Omega$ is moved to a feasible desired region $\hat{\Omega} \cap \Omega$ or a specific optimal point $\hat{\mathbf{x}}$.

III. Control and Design of Games

In this section, the game designer is assumed to have complete access to all game parameters and cost functions of players. Under this assumption, a static optimization approach to game design is briefly presented. This captures the cases when the game dynamics are very fast and the actions of the system designer are on a slower time-scale than the actual game dynamics between the players. Subsequently, a dynamic control approach is investigated for the case when the game dynamics are slow or there are external disturbances. Hence, the game is treated as a control system that needs to be stabilized around a desired point.

A. Static Optimization

When is it feasible to design a game such that the NE point can be located by the system designer to a point or region with desirable properties? In the point case, let target point be $\hat{\mathbf{x}}$. Then, the problem is to find an $\hat{\alpha}$ such that $\hat{\alpha} = \hat{\mathcal{T}}(\hat{\mathbf{x}})$, for any desirable feasible $\hat{\mathbf{x}}$. The following surprisingly simple result presented first in [21] addresses this problem for a broad class of games.

Theorem III.1. *For games of class $\mathcal{G}2$ with the cost structure given in (3) and under complete information assumption, affine pricing of the form, $\alpha\, p(\cdot)$, is sufficient to locate the unique NE point of the game to any desirable feasible point, $\hat{\mathbf{x}} \in \Omega$, as long as*

$$\frac{\partial p_i(\hat{\mathbf{x}})}{\partial x_i} \neq 0, \quad \forall i.$$

Proof. The proof immediately follows from the first order necessary optimality conditions of player cost optimization problems due to the convexity of the cost structure and uniqueness of NE.

$$\alpha_i \frac{\partial p_i(\hat{\mathbf{x}})}{\partial x_i} - \frac{\partial U_i(\hat{\mathbf{x}})}{\partial x_i} = 0 \Rightarrow \hat{\alpha}_i = \left[\frac{\partial p_i(\hat{\mathbf{x}})}{\partial x_i}\right]^{-1} \frac{\partial U_i(\hat{\mathbf{x}})}{\partial x_i} \quad \forall i$$

for any feasible $\hat{\mathbf{x}}$. □

Although tragedy of commons or price of anarchy are unavoidable in "pure" games without any external factors, they can be circumvented altogether when additional mechanisms such as "pricing" are included in the game formulation. In parallel to some earlier results [23], Theorem III.1 clearly establishes that *"loss of efficiency" or "price of anarchy"* are not an inherent feature of a broad class of games with built-in pricing systems. If there is sufficient information, then any game of class $\mathcal{G}2$ can be designed through simple (e.g. linear) pricing mechanisms in such a way that any desirable criteria such as welfare maximization or QoS requirements are met at the unique NE solution.

B. Dynamic Control of Games

Many games are solved in a distributed manner. However, the convergence of the system trajectory to the equilibrium point may not be very fast, hence the time-scale separation between system designers actions and actual game dynamics may fail. Then, the game design can be modeled as a **feedback control system** which utilizes pricing as the control input and the desired target as the reference (see Figure 2). This formulation also brings a certain robustness with respect to initial conditions or game (system) parameters. The latter case is especially relevant for systems that are non-stationary over longer time periods and can also be formulated as a tracking problem. Specific examples are congestion and power control game formulations, [9], [16], [22].

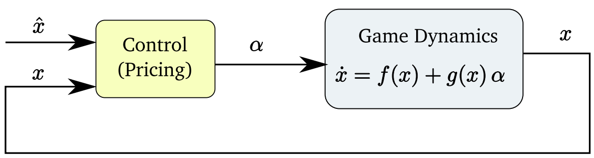

Fig. 2. Feedback control of the game (NE, $\mathbf{x}^*$) using pricing α as the control parameter and $\hat{\mathbf{x}}$ as the desired reference signal.

The counterpart of the feasibility question in the case of static game optimization relates in the dynamic control setting to the **controllability** of the system shown in Figure 2, or reachability of a state $\hat{\mathbf{x}}$. In order to provide a concrete example to the problem of controllability, consider a game of class $\mathcal{G}2$ where the players adopt a gradient algorithm to optimize their own cost. Then, the game dynamics are:

$$\dot{x}_i = -\frac{\partial J_i(\mathbf{x})}{\partial x_i} = \frac{\partial U_i(\mathbf{x})}{\partial x_i} - \frac{\partial p_i(\mathbf{x})}{\partial x_i}\,\alpha_i \quad \forall i, \tag{4}$$

where α acts as the feedback control on the outcome of the game. Here, the objective is to investigate the conditions under which the game system is controllable. We write (4) in vector form as

$$\dot{\mathbf{x}} = f(\mathbf{x}) + \sum_{i=1}^{N} g_i(\mathbf{x})\,\alpha_i = f(\mathbf{x}) + g(\mathbf{x})\,\alpha \tag{5}$$

where $\alpha = [\alpha_1 \dots \alpha_N]^T$, $g(\mathbf{x}) = [g_1(\mathbf{x}), \dots, g_N(\mathbf{x})]$ and

$$f(\mathbf{x}) = \begin{bmatrix} \frac{\partial U_1(\mathbf{x})}{\partial \mathbf{x}_1} & \dots & \frac{\partial U_i(\mathbf{x})}{\partial x_i} & \dots & \frac{\partial U_N(\mathbf{x})}{\partial \mathbf{x}_N} \end{bmatrix}^T$$

$$g(\mathbf{x}) = \begin{bmatrix} -\frac{\partial p_1(\mathbf{x})}{\partial \mathbf{x}_1} & \dots & 0 \\ \dots & -\frac{\partial p_i(\mathbf{x})}{\partial x_i} & \dots \\ 0 & \dots & -\frac{\partial p_N(\mathbf{x})}{\partial \mathbf{x}_N} \end{bmatrix}$$

Based on the standard theorem on controllability using Lie brackets [24, Chapter 1], we obtain the following result.

Theorem III.2. *For games of class $\mathcal{G}2$ with cost structure (3) and game dynamics (4), or (5), a sufficient condition for local reachability around a point $\mathbf{x}_0$ is that the distribution $\mathcal{C}$ satisfies the rank condition at $\mathbf{x}_0$, $dim\mathcal{C}(\mathbf{x}_0) = N$ where*

$$\mathcal{C} = \left[g_1, \dots, g_N, [g_i, g_j], \dots, [f, g_i], \dots, ad_f^k g_i, \dots\right]$$

where $[f, g](\mathbf{x}) = \frac{\partial g(\mathbf{x})}{\partial \mathbf{x}} f(\mathbf{x}) - \frac{\partial f(\mathbf{x})}{\partial \mathbf{x}} g(\mathbf{x})$ is the Lie bracket of f and g, and $ad_f^k g$ denote higher order Lie brackets defined recursively by $ad_f^k g(\mathbf{x}) = [f, ad_f^{k-1} g](\mathbf{x})$.

Remark III.3. Notice that if the diagonal matrix $g(\mathbf{x}_0)$ has rank N, any $\hat{\mathbf{x}}$ locally around $\mathbf{x}_0$ is reachable in finite time under piecewise constant input functions, which is equivalent to the feasibility condition in Theorem III.1. In addition, for the simple linear pricing function $p(x_i) = x_i$ any $\hat{\mathbf{x}}$ is immediately reachable since $g(\hat{\mathbf{x}})$ is constant and invertible.

Alternatively, the problem can be posed as one of asymptotic set point regulation, i.e., to find a feedback control of the form $\alpha = \alpha(\mathbf{x}, \hat{\mathbf{x}})$ such that the system trajectory $\mathbf{x}$ (and eventually the NE) converges to the desired reference point $\hat{\mathbf{x}}$ [24, Chapter 8]. Considering the game system (5) and the change of variables $\tilde{\mathbf{x}} = \mathbf{x} - \hat{\mathbf{x}}$, in the new coordinates the game system becomes

$$\dot{\tilde{\mathbf{x}}} = f(\tilde{\mathbf{x}} + \hat{\mathbf{x}}) + g(\tilde{\mathbf{x}} + \hat{\mathbf{x}})\,\alpha \tag{6}$$

and the design problem is to find control α to stabilize the equilibrium $\tilde{\mathbf{x}} = 0$. The first necessary condition, which translates into feasibility condition for $\hat{\mathbf{x}}$ is that (6) has an equilibrium at the origin, i.e. there exists a steady-state control $\alpha_s = c(\hat{\mathbf{x}})$ that solves

$$0 = f(\hat{\mathbf{x}}) + g(\hat{\mathbf{x}})\,c(\hat{\mathbf{x}}) \tag{7}$$

The component α_s is the first component in α needed to maintain equilibrium at the origin, while a second component is needed to asymptotically stabilize this equilibrium in the first approximation. Thus, based on the necessary and sufficient conditions for asymptotic regulation problem (see Theorem 8.3.2 in [24]), specialized for constant reference, a feedback control law for the pricing parameter that solves this problem in the full information case is $\alpha = \alpha(\mathbf{x}, \hat{\mathbf{x}})$ with

$$\alpha(\mathbf{x}, \hat{\mathbf{x}}) = c(\hat{\mathbf{x}}) + K\,(\mathbf{x} - \hat{\mathbf{x}}) \tag{8}$$

where $c(\hat{\mathbf{x}})$ solves (7) and K selected such that $(A + B\,K)$ has eigenvalues with negative real part. The matrices A and B denote the Jacobians of f and g at the origin, respectively. Using (5) it can be seen that the feasibility condition (7) is equivalent to feasibility in the static case (Theorem III.1). Also, for $g(\hat{\mathbf{x}})$ full rank this feasibility condition of the regulation problem corresponds to $\hat{\mathbf{x}}$ being reachable (see remark after Theorem III.2). Note also, that for constant reference $\hat{\mathbf{x}}$, integral control could be included in the design formulation by augmenting the system (5) with a stack of N integrators

$$\dot{\sigma} = \tilde{\mathbf{x}} \tag{9}$$

The task is to design a stabilizing feedback controller $\alpha = \alpha(\tilde{\mathbf{x}}, \sigma)$ that stabilizes the augmented state model (6,9) at the equilibrium point $(0, \sigma_s)$, where σ_s produces the desired steady-state control [24].

IV. Game Design under Information Constraints

Unlike the case discussed in the previous section, the system designer usually does not have full information about the system parameters such as user preferences or utility functions. Under such **information constraints**, the designer either deploys additional dynamic feedback mechanisms or requires side information from the system, depending on the specific design objectives. An example for the former case is a dynamic pricing scheme operating as an "outer feedback loop". If the objective is to achieve a social optimum (e.g. maximization of sum of user utilities) or satisfying some QoS constraints, then the designer often needs *accurate* side information from users or the system. Assuming that users are non-manipulative (honest or price-taking) and given accurate side information, the task of the designer can be formulated as an optimization problem even if it is solved indirectly or in a distributed manner. Next, an example formulation is provided that illustrates the underlying concepts. T*he objective is let the NE coincide with a social optimum (maximizing sum of user utilities) in the general case of non-separable user utilities of the form $U_i(\mathbf{x})$.*

A. Pricing Dynamics under Time-Scale Separation

Define a strictly concave and smooth social welfare function $\mathcal{U}(\mathbf{x})$ which is a sum of concave and **non-separable utility** functions $\mathcal{U}(\mathbf{x}) := \sum_i U_i(\mathbf{x})$ and admits a global maximum $\hat{\mathbf{x}} = \arg\max_{\mathbf{x}} \sum_i U_i(\mathbf{x})$. As before the cost function is $J_i(\alpha_i, \mathbf{x}) = \alpha_i p_i(\mathbf{x}) - U_i(x_i)$. Here, unlike the separable one in [21], side information (e.g. $U_i(\mathbf{x}^*)$) is required in order to bring the NE to the social maximum point. The social maximum is defined easily via the first order optimality conditions

$$\frac{\partial \mathcal{U}}{\partial \mathbf{x}}(\hat{\mathbf{x}}) = 0,$$

where

$$\frac{\partial \mathcal{U}}{\partial \mathbf{x}}(\mathbf{x}) = \left[\begin{array}{ccc} \sum_j \frac{\partial U_j}{\partial x_1}(\mathbf{x}) & \dots & \sum_j \frac{\partial U_j}{\partial x_N}(\mathbf{x}) \end{array} \right].$$

The social maximum is shown to coincide with the unique equilibrium (and NE) of the following pricing mechanism

$$\dot{\alpha}_i = \sum_j \sum_k \frac{\partial U_j}{\partial x_k^*} \frac{\partial x_k^*}{\partial \alpha_i} \quad \forall i \tag{10}$$

If these pricing dynamics are on a slower time scale than the game dynamics, then the system designer can obtain sufficiently accurate estimates of $\partial U_i(\mathbf{x}^*)/\partial x_i$ and $\partial x_i^*/\partial \alpha_i$. As one possibility, if the users adopt a gradient algorithm, then the designer can use past values of $\mathbf{x}^*$ and α along with the exact form of the pricing functions p to estimate $\partial U_i(\mathbf{x}^*)/\partial x_i$ directly without requiring any side information (except from some fixed system parameters). In addition, side information (e.g. $U_i(\mathbf{x}^*)$) is required to estimate $\partial U_i(\mathbf{x}^*)/\partial x_j$ and $\partial x_i^*/\partial \alpha_j$ for all i, j.

Assume an ideal case where the parameter estimation is perfectly accurate. Then, the pricing mechanism above ensures that the NE point of the underlying game globally asymptotically converges to the maximum of the social welfare function.[1] The next theorem follows from Lyapunov theory and LaSalle's theorem in a straightforward manner. The Lyapunov function is chosen to be the negative of social welfare function, $V = -\mathcal{U}$.

Assume that the pricing mechanism is on a slower time scale than the actual game dynamics leading to a time-scale separation, and hence to a hierarchically structured problem. Assuming this **time-scale separation**, for simplicity, initially only the pricing dynamics (slower dynamics) is considered.

Theorem IV.1. *Define an objective function $\mathcal{U}(\mathbf{x}) := \sum_i U_i(\mathbf{x})$ which admits a unique inner global maximum $\hat{\mathbf{x}} = \arg\max_{\mathbf{x}} \mathcal{U}(\mathbf{x})$ under suitable assumptions for user utilities U_i $\forall i$ in a class $\mathcal{G}2$ game. Then, the pricing mechanism (10) ensures that the NE point of the underlying game, $\mathbf{x}^*$, globally asymptotically converges to the maximum of the social welfare function, $\hat{\mathbf{x}}$, if the Jacobian matrix of the mapping $\mathcal{T}$ with respect to pricing vector α, defined as*

$$H(\alpha) := \frac{\partial \mathbf{x}^*}{\partial \alpha}(\alpha) = \left[\frac{\partial x_i^*}{\partial \alpha_j}(\alpha) \right], \;\; i, j = 1, \dots, N,$$

is non-singular.

The proof, which is omitted due to space constraints, focuses on the analysis of pricing (slow) dynamics assuming that the user dynamics are fast and converge quickly. It utilizes the Lyapunov theory and the LaSalle's theorem, (Theorem 4.4, [25]) to establish the result.

Remark IV.2. As an example, consider linear pricing and a signal-to-interference ratio (SIR)-like utility function

$$J_i(\alpha_i, \mathbf{x}) = \alpha_i \left(\sum_i x_i \right) - \beta_i \log(1 + s_i(\mathbf{x})),$$

where
$$s_i(\mathbf{x}) := \frac{h_i x_i}{\sum_{j \neq i} h_j x_j + \sigma}.$$

Then, it follows that under non-singularity conditions on the system matrix M, the unique NE is given as $\mathbf{x}^* = M^{-1}\,\mathbf{v}$ where $\mathbf{v} = [v_i]$, $v_i = \frac{\beta_i}{\alpha_i} - \sigma$. In this case x_i^* depends on all pricing parameters, α, and $H(\alpha)$ is not diagonal. However, we can still explicitly find $\partial x^*/\partial \alpha$, and it can be shown that, under non-singularity conditions on M^T, H is non-singular.

B. Game and Pricing Dynamics on Two Time-Scales

The previous analysis has focused on pricing dynamics under the time-scale separation, where game dynamics are assumed to be sufficiently fast. Removing this assumption for a complete treatment, two loops (on two time-scales) will be considered: one outer loop for pricing ($\dot{\alpha_i}$) and an inner loop for actions ($\dot{x_i}$). The next result presents the full analysis taking both user and pricing dynamics into account and is based on a **singular perturbation** or boundary layer

[1] For simplicity, the social maximum point is implicitly assumed to be on the solution space of the game.

approach, [25]. Towards this end, Theorem IV.1 is extended to analyze both pricing (slow) dynamics and user (fast) dynamics by using a combined Lyapunov function, [25].

Theorem IV.3. *Define an objective function* $\mathcal{U}(\mathbf{x}) := \sum_i U_i(\mathbf{x})$ *which admits a unique inner global maximum* $\hat{\mathbf{x}} = \arg\max_{\mathbf{x}} \mathcal{U}(\mathbf{x})$ *under suitable assumptions for user utilities* U_i $\forall i$ *in a class* $\mathcal{G}2$ *game. Then, under the pricing mechanism (10) the user dynamics (4) globally asymptotically converges to the maximum of the social welfare function,* $\hat{\mathbf{x}}$*, if the two systems are on separate time-scales, the Jacobian matrix of the mapping* $\mathcal{T}$ *with respect to pricing vector* α*,*

$$H(\alpha) = \frac{\partial \mathbf{x}^*}{\partial \alpha}(\alpha) = \left[\frac{\partial x_i^*}{\partial \alpha_j}(\alpha)\right], \quad i, j = 1, \ldots, N,$$

is non-singular and the Jacobian matrix Θ *of* $\dfrac{\partial J_i}{\partial x_i}$*, (4), with respect to* $\mathbf{x}$

$$\Theta(\alpha, \mathbf{x}) = \left[\frac{\partial^2 J_i(\alpha, \mathbf{x})}{\partial x_j \partial x_i}\right], \quad i, j = 1, \ldots, N,$$

is positive definite.

The proof of the theorem is omitted due to lack of space.

C. Illustrative Example

Consider the power control problem in optical networks [19] with linear pricing and optical signal-to-noise ratio (OSNR)-like utility and the following cost function

$$J_i(\mathbf{x}) = \alpha_i x_i - \beta_i (\log\left(1 + a_i \frac{\gamma_i(\mathbf{x})}{1 - \Gamma_{ii}\gamma_i(\mathbf{x})}\right) - x_i) \quad (11)$$

as a special case of (3). Here, β_i and a_i are design parameters, and $\gamma_i(\mathbf{x})$ denotes the OSNR

$$\gamma_i(\mathbf{x}) = \frac{x_i}{n_0 + \sum_j \Gamma_{ij} x_j} \quad (12)$$

for a given system matrix Γ and input noise n_0. Such a utility, i.e. the second term in (11) which may arise due to specific constraints on individual channel power, ensures the existence of a unique maximum for $\mathcal{U}(\mathbf{x}) = \sum_i U_i(x)$.

The Nash equilibrium $\mathbf{x}^*$ can be obtained from $\partial J_i / \partial x_i = 0$. If a_i are selected such that the following diagonal dominance condition holds

$$a_i > \sum_{j \neq i} \Gamma_{ij} \quad (13)$$

then $\mathbf{x}^*$ is explicitly expressed as $\mathbf{x}^* = \tilde{\Gamma}^{-1} C(\alpha)$, where

$$\tilde{\Gamma}_{ij} = \begin{cases} a_i, & i = j \\ \Gamma_{ij}, & i \neq j \end{cases},$$

and $C(\alpha)$ is a vector with elements $\frac{a_i \beta_i}{\alpha_i + \beta_i} - n_0$. Hence,

$$H(\alpha) = \frac{\partial \mathbf{x}^*}{\partial \alpha}(\alpha) = \tilde{\Gamma}^{-1} \mathrm{diag}\left(\frac{-\mathrm{a_i}\beta_\mathrm{i}}{(\alpha_\mathrm{i} + \beta_\mathrm{i})^2}\right) \quad (14)$$

is clearly non-singular. The second condition in Theorem IV.3 is that $\Theta > 0$, where Θ is for this example:

$$\Theta_{ii} = \frac{\partial^2 J_i}{\partial x_i^2} = \frac{a_i^2 \beta_i}{(n_0 + \sum_j \tilde{\Gamma}_{ij} x_j)^2}$$

$$\Theta_{ij} = \frac{\partial^2 J_i}{\partial x_j \partial x_i} = \frac{a_i \beta_i \Gamma_{ij}}{(n_0 + \sum_j \tilde{\Gamma}_{ij} x_j)^2}.$$

It can be immediately seen that if (13) holds, then the matrix Θ is strictly diagonal dominant. If in addition, $a_i > \sum_{j \neq i} \Gamma_{ji}$ it can be shown that Θ^T is strictly diagonal dominant, hence Θ is positive definite. The closed loop system (10), (4) is

$$\dot{\alpha} = H^T(\alpha) \left[\frac{\partial \mathcal{U}}{\partial \mathbf{x}}(\mathbf{x})\right]^T \quad (15)$$

$$\dot{\mathbf{x}} = \bar{f}(\mathbf{x}) - \alpha \quad (16)$$

where $H^T(\alpha)$ is defined in (14), the j^{th} element in $\dfrac{\partial \mathcal{U}}{\partial \mathbf{x}}(\mathbf{x})$, for $\mathcal{U}(\mathbf{x}) := \sum_i U_i(\mathbf{x})$, is given as

$$\frac{\partial \mathcal{U}}{\partial x_j}(\mathbf{x}) = \bar{f}_j(\mathbf{x}) - X_j(\mathbf{x}) \quad (17)$$

with

$$\bar{f}_j(\mathbf{x}) = \frac{a_j \beta_j}{n_0 + \sum_k \tilde{\Gamma}_{jk} x_k} - \beta_j$$

$$X_j(\mathbf{x}) = \sum_{p \neq j} \frac{a_p \beta_p \Gamma_{pj} x_p}{(n_0 + \sum_{k \neq p} \Gamma_{pk} x_k)(n_0 + (\sum_k \tilde{\Gamma}_{pk} x_k)}$$

Thus (15), (16) represent the closed-loop dynamic system. The objective function $\mathcal{U}$ satisfies $\partial^2 \mathcal{U} / \partial^2 x < 0$ for a nonempty set of the design variables a, β, but the conditions are very complicated and as such the details are omitted. The following section presents an example based on realistic parameters for the two channel case.

V. Simulations

In this section, we provide simulation results for the example in Section IV.C, (15), (16) for the two players case. We consider an optical fiber link with ten amplifiers, each with a parabolic gain shape according to the formula $G = -4e16 \times (\lambda - 1555 \times 10^{-9})^2 + 15$ dB, where λ denotes a channel wavelength, and a span loss of 10 dB. The system matrix Γ is obtained as

$$\Gamma = \begin{bmatrix} 2.47 \times 10^{-3} & 2.61 \times 10^{-3} \\ 2.36 \times 10^{-3} & 2.5 \times 10^{-3} \end{bmatrix}$$

The following parameters are used: $\beta_i = 1$, $a = [0.485,\ 0.48]$ such that they satisfy the diagonal dominance condition on Γ and $\partial^2 \mathcal{U} / \partial^2 x < 0$. These parameters yield the equilibrium solution $\hat{\mathbf{x}} = [0.0134,\ 0.0128]$ milliwatt (mW) and $\hat{\alpha} = [73.4,\ 76.9]$. An input noise power of $n_0 = 0.43$ nanowatt (nW) is considered. The closed-loop system (15), (16) is simulated for $\epsilon = 0.01$, i.e., on two time scales such that in discrete-time form the pricing algorithm (15) is run every 100 iterations of the user algorithm (16). The user algorithm is implemented in

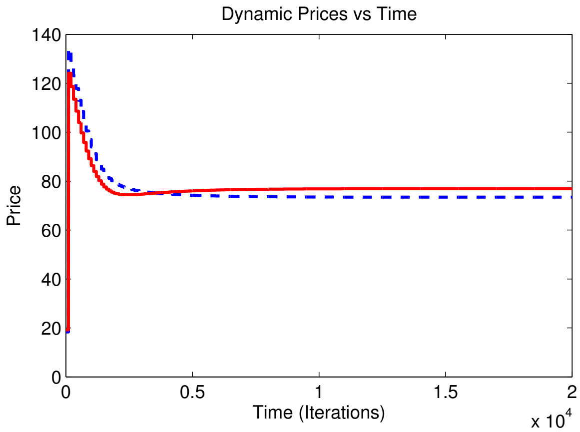

Fig. 3. Channel prices, α, as a function of time.

a decentralized way such that only the explicitly measurable OSNR γ_i signal is fed back to the respective channel source.

The simulations, which take $\alpha = [18.35 \;\; 19.23]$ and $\mathbf{x} = [0.00043 \;\; 0.00043]^T$ as initial conditions and run for 50 (5000) iterations of the pricing (users) algorithm, show a clear convergence to the equilibrium solution $\hat{\mathbf{x}}$, $\hat{\alpha}$ and to the OSNR values of approximately 23dB. Figure 3 depicts the evolution of channel prices, α, as a function of time.

VI. Conclusion

A control theoretic approach to design of noncooperative games is presented. It is shown that for a broad class of noncooperative games, pricing mechanisms can be used as a design tool to control the outcome of the game such that certain predefined global objectives are satisfied. The class of games studied are widely applicable to a variety of problems, such as network congestion control, wireless uplink power control, and optical power control. The game design problem is analyzed under full and limited information assumptions for dynamic systems and non-separable utility functions. Sufficient conditions are derived, which allow the designer to place the Nash equilibrium solution or to guide the system trajectory to a desired region or point. The results obtained are illustrated through examples.

There are many directions for extending the results presented. One of them includes further application of game design methods to specific problems such as power control in optical networks and spectrum allocation in wireless networks. Another interesting direction is the analysis of estimation error effects on mechanism stability for the limited information case. Furthermore, the stability of the combined pricing and game system can be analyzed under time delays which often have a nonnegligible effect on stability.

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