Randomized Algorithms for Stability and Robustness Analysis of High Speed Communication Networks

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***Abstract*— This paper initiates a study toward developing and applying randomized algorithms for stability of high speed communication networks. We consider the discrete-time version of the nonlinear model introduced in [1], which uses as feedback variations in queueing delay information from bottleneck nodes of the network. We then linearize this nonlinear model around its unique equilibrium point at a single bottleneck node, and perform a robustness analysis for a special, *symmetric* case, where certain utility and pricing parameters are the same across all active users. In this case, we derive closed-form necessary and sufficient conditions for stability and robustness under parameter variations. In addition, the ranges of values for the utility and pricing parameters for which stability is guaranteed are computed exactly. These results also admit counterparts for the case when the pricing parameters vary across users, but the utility parameter values are still the same. In the general non-symmetric case, when closed-form derivation is not possible, we construct specific randomized algorithms which provide a probabilistic estimate of the local stability of the network. In particular, we use Monte Carlo as well as Quasi-Monte Carlo techniques for the linearized model. The results obtained provide a complete analysis of congestion control algorithms for internet style networks with a single bottleneck node as well as for networks with general random topologies.**

I. Introduction

High speed communication networks recently received growing attention in the control literature, as evidenced by the appearance of several special issues devoted to this topic in leading journals in the field, such as [2], [3] and [4]. Various approaches and solutions have been developed and studied in this context, including modeling of TCP/IP traffic, congestion control for ABR service in ATM networks, packet marking schemes for the Internet, application of low order controllers for AQM, as well as related problems.

One of the critical issues that lie at the heart of efficient operation of high speed networks is *congestion control*. This involves the problem of regulating the source rates in a decentralized and distributed fashion, so that the available bandwidths on different links are used most efficiently while minimizing (or totally eliminating) loss of packets due to queues at buffers exceeding their capacities. This objective has to be accomplished under variations in network conditions such as packet delays (due to propagation as well as queueing) and bottleneck nodes. This paper addresses this challenge using *randomized algorithms* within the context of a model introduced in [1], which is based on noncooperative game theory [12], and captures all the elements and features mentioned above. The original model is nonlinear and in continuous time (CT), but we work here with a discrete-time (DT) version of it as any implementation of the CT model will inevitable involve a discretization synchronized with the round trip time (RTT) of packets. The DT model is also nonlinear, and depends on a number of parameters representing pricing, utility (to individual users), and link capacities. It has a unique equilibrium state for each set of values of these parameters, and the objective is to establish stability in a region of the parameter space taken as large as possible. The presence of the nonlinearities in the DT model makes it impossible to obtain a global stability result (even though this is possible for the CT version [1]), which forces us to study the linearized version around the equilibrium state, which is also of independent interest. The goal, then restated, is to establish local stability and robustness to parameter variations of the DT model.

However, even this goal cannot be accomplished analytically, due to nonlinear dependence of the linearized model on the key system parameters, except in some special cases. One such scenario is the so-called *user-symmetric* case, which corresponds to the situation when certain utility network parameters are all equal. For the more general case, one has to resort to other (non-analytic) tools, among which the approach using *randomized algorithms* ([6], [7]) stands out as a strong contender.

The study of randomized algorithms for analysis and design of control systems has aroused considerable interest in the systems and control community. Randomized algorithms are efficient and low complexity, and are useful especially when worst case analysis of complex systems is either very difficult or impossible. Unlike more classical methods, these algorithms yield an assessment on the satisfaction of required specifications with a certain probabilistic accuracy. They provide an alternative solution with a trade-off between computational complexity and tightness of the solution.

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Randomized algorithms rely heavily on univariate and multivariate methods for sample generations in various sets [5]. Roughly speaking, sample generation techniques can be divided into two main categories: Monte Carlo (MC), see e.g. [14] and Quasi-Monte Carlo (QMC) [15]. While the former is classical, statistical-based, and assumes an *a priori* knowledge of probability density functions, the latter may be regarded as its deterministic counterpart. The main objective of QMC methods is to reduce the "discrepancy" between the generated samples, and a secondary objective is to avoid the curse of dimensionality that arises in gridding or rejection methods. Even though QMC methods may be more suitable than MC algorithms in certain situations, a definitive conclusion is not yet available. One of the contributions of this paper is to provide a comparison between these methods for networking problems–an unexplored domain for randomized algorithms.

The paper is organized as follows. In Section 2 we present the general network model and the congestion control algorithm. An analytical local stability analysis of the single bottleneck node case with symmetric users is given in Section 3. In Section 4, we give a brief account of randomized algorithms and discuss their use in the present context. In Section 5, numerical results are presented for stability of the linearized system first for a single bottleneck node and subsequently under general network topologies. Conclusions are then provided in Section 6.

II. The Model

A. The Network Model

We consider a somewhat simplified version of a general network model based on fluid approximations introduced in [1]. The topology of the network studied here is characterized by a set of nodes $\mathcal{N} = \{1, \ldots, N\}$ and a set of links $\mathcal{L} = \{1, \ldots, L\}$, with each link $l \in \mathcal{L}$ having a fixed capacity $C_l > 0$, and an associated buffer size $b_l \geq 0$. There are M users, with the set of users being $\mathcal{M} := \{1, \ldots, M\}$. Each user is associated with a unique *connection*, or path R, between a source and a destination node. The nonnegative flow, x_i, sent by the i-th user over this path R_i satisfies the bounds $0 \leq x_i \leq x_{i_{\max}}$. We define a routing matrix, $\mathbf{A}$, of ones and zeros as in [13], which describes the relation between the set of routes $\mathcal{R}$ associated with the users (connections) and links. We then have the capacity constraint $\mathbf{A}\mathbf{x} \leq \mathbf{C}$, where $\mathbf{x}$ is the $(M \times 1)$ flow rate vector of the users and $\mathbf{C}$ is the $(L \times 1)$ link capacity vector. If the aggregate sending rate of users whose flows pass through link l exceeds the capacity C_l of the link, then the arriving packets are queued (generally on a first-come first-serve basis) in the buffer b_l of the link, which evolves according to $\dot{b}_l(t) = \sum_{i:l \in R_i} x_i(t) - C_l$.

B. System Dynamics and the Cost Function

The cost function makes use of the variation in queueing delay d, defined as the difference between the actual delay d^a experienced by a packet and the propagation delay d^p of the connection [1]. The variation in queueing delay is an important indication of congestion for Internet-style networks, and it can be modeled for link l as $\dot{d}_l(\mathbf{x}, t) := \frac{\partial d_l}{\partial t} = \frac{1}{C_l} \left(\sum_{i:l \in R_i} x_i(t) - C_l \right)$, where $\sum_{i:l \in R_i} x_i$ is the total flow on the link. Thus, the queueing delay that a user experiences is the sum of queueing delays on its path, that is $D_i(\mathbf{x}, t) = \sum_{l \in R_i} d_l(\mathbf{x}, t)$. The cost function for the i-th user at time t is the difference between a linear pricing function proportional to the queueing delay the user experiences and a strictly increasing logarithmic utility function multiplied by a user preference parameter u_i

$$J_i(\mathbf{x}, t) = \alpha_i \, D_i(\mathbf{x}, t) \, x_i \, - u_i \log(x_i + 1). \tag{1}$$

The users pick their flow rates so as to minimize their cost functions (and in this sense we have a noncooperative game). Consistent with this goal we adopt a dynamic update model whereby each user changes his flow rate proportional to the gradient of his cost function with respect to his flow rate. Thus, the algorithm for the i-th user is

$$\frac{dx_i}{dt} = \dot{x}_i = -\frac{\partial J_i(\mathbf{x})}{\partial x_i} = \frac{u_i}{x_i + 1} - \alpha_i \, D_i, \tag{2}$$

where we have ignored the effect of the i-th user's flow on the delay D_i that s/he experiences. This assumption can be justified for networks with a large number of users.

In a realistic implementation of the algorithm, the users update their flow rates only at discrete time instances corresponding to multiples of RTT, and hence we discretize the reaction function of the i-th user, and normalize it with respect to the RTT of the user. The optimal user response is, therefore, a discrete-time version of (2), and is given by, [1]

$$x_i(t+1) = x_i(t) + \kappa_i \left[\frac{u_i}{x_i(t) + 1} - \alpha_i \sum_{l \in R_i} d_l(t) \right], \quad t = 0, 1, \ldots, \;\; i \in \mathcal{M}, \tag{3}$$

where κ_i is a (user specific) step-size constant which will be taken to be 1 for the rest of the paper, which is no loss of generality since it can be absorbed into the other parameters u_i and α_i. Furthermore, we take $x_i(0) = 0\,,\; i \in \mathcal{M}$, without any loss of generality. The queue model is discretized in a similar manner, with the queueing delay at link l being

$$d_l(t+1) = d_l(t) + \frac{1}{C_l} \sum_{i:l \in R_i} x_i(t) - 1, \; t = 0, 1, \ldots, \tag{4}$$

with $d_l(0) = 0\,,\;\; l \in \mathcal{L}$.

III. Stability Analysis: The symmetric single bottleneck case

Let us consider the case of a single bottleneck node, with M users having connections passing through that node. Hence, we have essentially a single link of interest, for which we denote the associated delay by d, and likewise the associated capacity by C. Then, the equilibrium state of the system described by (3) and (4) follows readily as

$$x_i^* = \frac{u_i}{\alpha_i d^*} - 1\,,\; i \in \mathcal{M} \text{ and } \quad d^* = \frac{1}{C + M} \sum_{i=1}^{M} \frac{u_i}{\alpha_i}\,, \tag{5}$$

[1] We have abused the notation here, as t here does not correspond to the t in the CT description. Since the CT description will not be used in the rest of the paper, this should not create any ambiguity or confusion.

which is unique.

Let $\widetilde{x}_i(t) := x_i(t) - x_i^*$, $i \in \mathcal{M}$, and $\widetilde{d}(t) := d(t) - d^*$. The system (3)-(4) with a single bottleneck link and with $\kappa_i = 1$ can now be rewritten around the equilibrium state as

$$\widetilde{x}_i(t+1) = \widetilde{x}_i(t) + \frac{u_i}{\widetilde{x}_i(t) + x_i^* + 1} - \alpha_i(\widetilde{d}(t) + d^*),\ i \in \mathcal{M}$$
$$\widetilde{d}(t+1) = \widetilde{d}(t) + \frac{1}{C}\sum_{i=1}^{M} \widetilde{x}_i(t). \tag{6}$$

Let $\widetilde{\mathbf{x}}$ denote the M-dimensional column vector whose entries are the $\widetilde{x}_i$'s; likewise let the M-dimensional column vector whose entries are the $\widetilde{x}_i^*$'s be denoted by $\widetilde{\mathbf{x}}^*$. Linearizing (6) around $(\widetilde{\mathbf{x}}^*, \widetilde{d}^*) = (0, 0)$, we obtain

$$\widetilde{x}_i(t+1) = \left[1 - \frac{u_i}{(x_i^* + 1)^2}\right]\widetilde{x}_i(t) - \alpha_i \widetilde{d}(t)\,,\quad i \in \mathcal{M}$$
$$\widetilde{d}(t+1) = \widetilde{d}(t) + \frac{1}{C}\sum_{i=1}^{M} \widetilde{x}_i(t)\,. \tag{7}$$

Define $\alpha := [\alpha_1, \alpha_2, \ldots, \alpha_M]$ and $\beta := [\beta_1, \beta_2, \ldots, \beta_M]$, where

$$\beta_i := u_i/(x_i^* + 1)^2\,,\quad i \in \mathcal{M}\,. \tag{8}$$

The system equations (7) can then be expressed in matrix form

$$\begin{pmatrix} \widetilde{\mathbf{x}}(t+1) \\ \widetilde{d}(t+1) \end{pmatrix} = \mathbf{L}(\alpha, \beta, C) \begin{pmatrix} \widetilde{\mathbf{x}}(t) \\ \widetilde{d}(t) \end{pmatrix} \tag{9}$$

where

$$\mathbf{L}(\alpha, \beta, C) = \begin{pmatrix} 1-\beta_1 & 0 & \cdots & -\alpha_1 \\ 0 & 1-\beta_2 & & -\alpha_2 \\ \vdots & & \ddots & \vdots \\ \frac{1}{C} & \frac{1}{C} & \cdots & 1 \end{pmatrix}. \tag{10}$$

Hence the system (6) is locally asymptotically stable if and only if $\mathbf{L} = \mathbf{L}(\alpha, \beta, C)$ is *Schur*, that is all its eigenvalues, $\lambda(\alpha, \beta, C)$, are in the open unit circle. The goal now is to find the region in the parameter space (with the parameters being α, β, and C), such that $|\lambda(\alpha, \beta, C)| < 1$. We consider first the special case of symmetric users having the same pricing and utility preference parameters, that is $u_i = u$ and $\alpha_i = \alpha$ for all $i \in \mathcal{M}$, which also implies that $\beta_i = \beta$ for all $i \in \mathcal{M}$. When $M = 2$, one can explicitly determine the eigenvalues of the matrix $\mathbf{L}$

$$\lambda_1 = 1 - \beta$$
$$\lambda_{2,3} = 1 - \frac{\beta}{2} \pm \sqrt{\frac{\beta^2}{4} - 2\frac{\alpha}{C}} \tag{11}$$

which are in the open unit circle if and only if

$$\frac{2\alpha}{C} < \beta < 2\,.$$

The lemma below and the proposition that follows generalizes this result to M users.

Lemma III.1. *If the user preference parameters and prices are symmetric across all M users, that is $u_i = u$ and $\alpha_i = \alpha$, then the characteristic equation of the matrix $\mathbf{L}$ is given by*

$$\det(\lambda \mathrm{I} - \mathbf{L}) = (\lambda - (1-\beta))^{M-1} \left[\lambda^2 - (2-\beta)\lambda + 1 - \beta + \frac{M\alpha}{C}\right]. \tag{12}$$

Thus, $\mathbf{L}$ has $M-1$ real eigenvalues at $1-\beta$ and two, possibly complex, eigenvalues at

$$1 - \beta/2 \pm \sqrt{\beta^2/4 - M\alpha/C}\,.$$

We now determine the region in the parameter space where $\mathbf{L}$ is Schur matrix. It readily follows from the lemma that the condition $0 < \beta < 2$ is both necessary and sufficient for $M-1$ real roots $(1-\beta)$ to be in the open unit circle. On the other hand, the remaining two possibly complex roots of (12) have their absolute values strictly less than one, $|\lambda| < 1$, if and only if

$$\beta \in \left(\min\left\{\frac{M\alpha}{C}, 2\sqrt{\frac{M\alpha}{C}}\right\}, 2 + \frac{M\alpha}{2C}\right).$$

Combining this with the earlier condition $0 < \beta < 2$, we arrive at a necessary and sufficient condition for local stability of the equilibrium state of system (6) in the symmetric user case.

Proposition III.2. *If the user preference parameters and prices are symmetric across all M users (that is, $u_i = u$ and $\alpha_i = \alpha$, which further implies that $\beta_i = \beta$), the single bottleneck system (6) is locally stable around its equilibrium state (5) if and only if α, β, and C lie in the region $(M\alpha/C) < \beta < 2$.*

Remark III.3. If $C = \mu M$, for some positive constant μ, the necessary and sufficient condition becomes $(\alpha/\mu) < \beta < 2$.

The condition in Proposition III.2 can also be expressed in terms of u, together with α and C. First note the relationship

$$\beta = [M/(C+M)]u\,,$$

which follows from (5) and (8). In view of this relationship, we immediately have the following corollary to Proposition III.2.

Corollary III.4. *For the symmetric parameter case, the single bottleneck system (6) is locally stable around its equilibrium state (5) if and only if α, u, and C lie in the region*

$$(1 + \frac{M}{C})\alpha < u < 2\big(1 + \frac{C}{M}\big)\,.$$

Finally, we generalize Proposition III.2 by removing the symmetry in the pricing parameter α, while retaining the symmetry in β.

Proposition III.5. *Let β be symmetric across all M users, $\beta_i = \beta$, while $\alpha = [a_1, \ldots, a_M]$ is general. Then, the characteristic equation of the matrix $\mathbf{L}$ is given by*

$$\det(\lambda \mathrm{I} - \mathbf{L}) = (\lambda - (1-\beta))^{M-1} \big[\lambda^2 - (2-\beta)\lambda + 1 - \beta + \sum_{i=1}^{M}\frac{\alpha_i}{C}\big]. \tag{13}$$

The matrix $\mathbf{L}$ *has* $M-1$ *real eigenvalues at* $1-\beta$ *and two, possibly complex, eigenvalues at*

$$1-\frac{\beta}{2}\pm\sqrt{\frac{\beta^2}{4}-\sum_{i=1}^{M}\frac{\alpha_i}{C}}\ .$$

Furthermore, the single bottleneck link system (6) is locally stable around its equilibrium state (5) if and only if the parameters $\{\alpha_i\ ,\ i\in\mathcal{M}\}$*,* β*, and* C *lie in the region*

$$\frac{1}{C}\sum_{i=1}^{M}\alpha_i<\beta<2\,.$$

IV. Randomized Algorithms and Stability Analysis for the Non-Symmetric Case

We have seen in the previous section that local stability and robustness can be studied analytically (because the eigenvalues of $\mathbf{L}$ can be computed explicitly) when the u_i's are the same for all users (or equivalently when the β_i's are the same). If this is not the case, however, then the eigenvalues of $\mathbf{L}$ cannot be expressed in closed form, making it very challenging (if not impossible) to deduce any reasonable stability and robustness results using analytical techniques. Then, one has to resort to simulation-based methods, and as mentioned earlier *randomized algorithms* stand out as the most promising.

A. Monte Carlo (MC) Methods

In MC methods, the first step is to assume that the parameter vectors α and β are random with given probability density functions f_α and f_β, having support sets $\mathcal{B}_\alpha$ and $\mathcal{B}_\beta$, respectively. We can take, for example, $\mathcal{B}_\alpha$ and $\mathcal{B}_\beta$ to be the hyper-rectangular sets

$$\begin{aligned}\mathcal{B}_\alpha &= \{\alpha:\alpha_i\in[\alpha_i^-,\alpha_i^+]\,,\ i=1,2,\ldots,M\}\,,\\ \mathcal{B}_\beta &= \{\beta:\beta_i\in[\beta_i^-,\beta_i^+]\,,\ i=1,2,\ldots,M\}\,,\end{aligned}$$

and f_α and f_β to be uniform with support sets $\mathcal{B}_\alpha$ and $\mathcal{B}_\beta$, respectively. Then, we generate N_α independent identically distributed (i.i.d.) vector samples, $\alpha^1,\alpha^2,\ldots,\alpha^{N_\alpha}$, from the set $\mathcal{B}_\alpha$ according to f_α. Similarly, N_β i.i.d. vector samples, $\beta^1,\beta^2,\ldots,\beta^{N_\beta}$, from the set $\mathcal{B}_\beta$ are generated according to f_β. Subsequently, using (10) we compute $\mathbf{L}(\alpha^i,\beta^k)$ for $i=1,2,\ldots,N_\alpha$, and $k=1,2,\ldots,N_\beta$, where we have suppressed the dependence on C.

The next step is to construct the indicator function

$$\mathcal{I}(\alpha^i,\beta^k):=\begin{cases}1 & \text{if } \mathbf{L}(\alpha^i,\beta^k) \text{ is Schur}\\ 0 & \text{otherwise.}\end{cases}$$

The estimated *probability of stability* is readily given by

$$\widehat{p}_{N_\alpha,N_\beta}=\frac{1}{N_\alpha N_\beta}\sum_{i=1}^{N_\alpha}\sum_{k=1}^{N_\beta}\mathcal{I}(\alpha^i,\beta^k)$$

which is equivalent to $\widehat{p}_{N_\alpha,N_\beta}=\frac{N_{good}}{N_\alpha N_\beta}$, where N_{good} is the number of vector samples such that $\mathbf{L}(\alpha^i,\beta^k)$ is a Schur matrix. The estimate $\widehat{p}_{N_\alpha,N_\beta}$ is usually referred to as *empirical probability*.

Clearly, for a finite sample size, it is important to know how many samples N_α and N_β are needed to obtain a "reliable" probabilistic estimate $\widehat{p}_{N_\alpha,N_\beta}$. To this end, classical results, such as the Chernoff bound can be used; see [8] for details. It is important to remark, however, that the number of required vector samples is independent of the problem dimension, e.g. of the size of the matrix $\mathbf{L}(\alpha,\beta)$ and of the number of users, M. Another important issue in MC methods is the development of efficient algorithms for sample generation in various sets according to different distributions. In particular, the problem is how to efficiently generate N_α and N_β vector samples α^i and β^k according to the given densities f_α and f_β, and support sets $\mathcal{B}_\alpha$ and $\mathcal{B}_\beta$. For univariate density functions, this specific problem is equivalent to the one of generating uniform random numbers in the interval $[0,1]$. We refer to [10] and [11] for further details. In the case of multivariate distributions rejection methods can be used for general distributions and support sets (see e.g. [9] and [14]).

B. Quasi-Monte Carlo (QMC) Methods

In the case of QMC methods, the sample generation is obtained in a completely different way. That is, no probability density functions f_α and f_β are specified or used and the samples are generated according to a purely deterministic mechanism. Therefore, the sequences $\alpha^1,\alpha^2,\ldots,\alpha^{N_\alpha}$ and $\beta^1,\beta^2,\ldots,\beta^{N_\beta}$ are now quasi-random and are chosen in order to minimize the so-called discrepancy, which is a measure of how the sample set is "evenly distributed" within a given set.

Formally, the discrepancy $D(S,\mathcal{B})$ of a sample set $S\in\mathcal{B}$ of cardinality N is defined as [15]

$$D(S,\mathcal{B})=\sup_{B\in\mathcal{B}}\left|\frac{|S\cap B|}{N}-\mathrm{Vol}(B)\right|,$$

where B is any subset of $\mathcal{B}$, $\mathrm{Vol}(B)$ is the volume of B and $|S\cap B|$ denotes the cardinality of $S\cap B$. The idea is to "cover" the set $\mathcal{B}$ as uniformly as possible for a given sample size. One can ask, on the other hand, why a simple uniform grid providing low discrepancy is not preferred. Even though the apparent randomness of quasi-random sequences may be attractive for various reasons, the main benefit is to avoid the curse of dimensionality which is inherent to gridding techniques. That is, as the dimension of the parameter space increases, the number of samples required to cover the set $\mathcal{B}$ with a uniform grid grows exponentially. Various classical low discrepancy sequences are available in the literature, including Halton, Sobol, Niederreiter and others. Finally, we note that discrepancy is not the only criterion used. For example, the so-called *dispersion*, which is a normalized lower bound on the discrepancy is also studied in [15] and [16].

V. Simulation Results

Since the uncertainty in the general case consists of non-linearly coupled parameters even in the single bottleneck link case as shown in (10), we make use of randomized algorithms instead of classical worst case analysis. For the remainder of the paper, the term stability refers to probability of stability when MC methods are used and to the ratio of the number of successes over the number of experiments in the case of QMC methods.

A. Single Bottleneck Link with Multiple Users

We have seen earlier in Section III that we can study local stability and robustness in two different parameter spaces, namely (α, u, C) and (α, β, C), where the former admits an interpretation in terms of the original model, whereas the latter is just a transformation which was introduced for convenience. In any analytical study, such as the one in Section III, it does not make any difference whether one works with the former or the latter, since there is a one-to-one transformation between the two parameter spaces. In the case of randomized algorithms, however, it does make a difference, since the distribution one uses for one space does not necessarily correspond to the one used for the other. For this reason we carry out the analysis with randomized algorithms in both parameter spaces.

For the case of a single bottleneck link network, we first study local stability and robustness in the u-α parameter space, $p = [\alpha_1, \ldots, \alpha_M, u_1, \ldots, u_M, C]$. The matrix in question is (10), which is expressed in terms of β_i's, which however can be expressed in terms of u_i's through (8). Subsequently, we carry out the study in the α-β parameter space, where the p vector is now $p = [\alpha_1, \ldots, \alpha_M, \beta_1, \ldots, \beta_M, C]$. In both cases, we use not only MC methods as the probabilistic model for the system, but also QMC sequences like, Halton, Sobol, and Niederreiter in order to–presumably–obtain a better coverage of the $2M + 1$-dimensional parameter space.

1) The u-α parameter space: We first simulate the effect of bottleneck link capacity C on the local stability of the system for various values of u and α. In this simulation we use MC, QMC and grid methods together, which enables us to compare the performance of these methods. Although the grid method is the most reliable one as it covers the parameter space deterministically, it is prohibitive due to its computational complexity in higher dimensional systems. Due to this limitation, we simulate a system with 4 users only.

For all methods, the parameter ranges $0 < \alpha_i < 0.2$ and $0 < u_i < 20,000,\ i = 1, \ldots, 4$, are chosen with 100% tolerance around their nominal values. For the probabilistic model for parameters, we use a uniform distribution. On the unit interval $[0, 1]$, the grid is constructed through points spaced as $[0.125\ 0.375\ 0.625\ 0.875]$ in each dimension. The number of samples is chosen sufficiently large as $N = 65,536$. Results of this simulation are shown in Fig. 1. We observe that the system is locally stable only for a certain range of C.

We note that the specific implementations of the QMC algorithms that we use have technical limitations as dimension of the system increases. Hence, for a large number of users, we limit our analysis to MC methods only.

Finally, we investigate the robustness of the system with respect to various user and pricing parameters given a fixed capacity at the bottleneck link. For each case, the user and pricing parameters are uniformly distributed with up to 100% tolerance around their nominal values. We further take $C = 2,000,000$, $M = 20$, and $N = 100,000$. As a result of the simulation, we conclude that local stability decreases as nominal values of u and α increase.

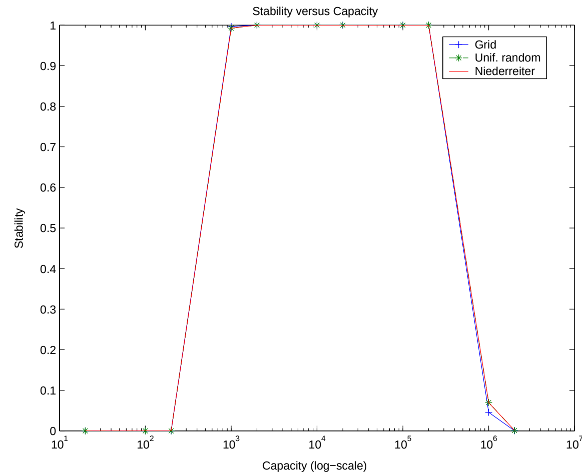

Fig. 1. Stability versus capacity (logarithmic scale) for $M = 4$ users using MC, QMC, and grid methods. Parameters have 100% tolerance around nominal values.

2) α-β parameter space: We now carry out the preceding analysis in the α-β parameter space. As noted earlier, the $\alpha-\beta$ space is a nonlinear transformation of the u-α space, and hence any sample distribution in the former corresponds to some other sample distribution in the latter. We first look at the effect of capacity through a system with 4 users. For all the methods, the parameter ranges are taken to be $0 < \beta_i < 1$ and $0 < \alpha_i < 1,000,\ i = 1, \ldots, 4$. As the probabilistic model for parameters a uniform distribution is chosen. The grid is constructed through points spaced as $[0.1\ 0.3\ 0.5\ 0.7\ 0.9] \times R_{max}$ in each dimension, where R_{max} is 1 for β and $1,000$ for α. Furthermore, $N = 390,625$. Results of this simulation are shown in Fig. 2, where we observe that the stability of the system improves as C increases.

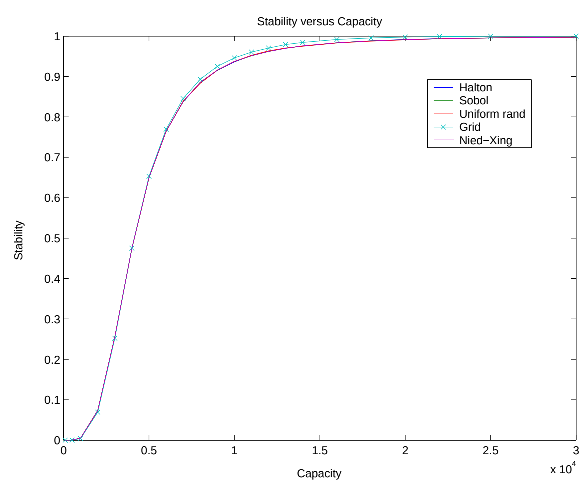

Fig. 2. Stability versus capacity for $M = 4$ users using MC, QMC, and grid methods. Parameter ranges are $0 < \beta_i < 1$ and $0 < \alpha_i < 1,000$, $i = 1, \ldots, 4$.

Next, we investigate robustness of the system with respect to a range of parameters (α-β) using uniform random distribution within the ranges $[0,1]$ for β and $[0,1000]$ for α. We choose $C = 25,000$, $M = 20$, and $N = 100,000$. We observe that local stability degrades as the ranges of β_i's and α_i's increase.

B. General Network Topology

We now turn our attention to local stability and robustness of a general topology network with multiple bottleneck links, and routing matrix $\mathbf{A}$. The system equations are given in (3) and (4). For this general case, equilibrium point or points of the system cannot be described explicitly. However, the uniqueness of the equilibrium state can be established using the following proposition, which can be found in [1]:

Proposition V.1. *When* $\mathbf{A}$ *is full row rank, the system described by (3) and (4) has a unique equilibrium.*

Let $\widetilde{x}_i(t) := x_i(t) - x_i^*$ and $\widetilde{d}_l(t) := d_l(t) - d_l^*$, given the existence of a unique equilibrium point, $(\mathbf{x}^*, \mathbf{d}^*)$. Linearizing the system (3)-(4), with $\kappa = 1$, around $(\tilde{\mathbf{x}}^*, \tilde{d}^*) = (0,0)$, we obtain

$$\begin{aligned} \widetilde{x}_i(t+1) = \quad & [1 - \frac{u_i}{(x_i^*+1)^2}]\widetilde{x}_i(t) - \alpha_i \sum_{l \in R_i} \widetilde{d}_l(t), \\ \widetilde{d}_l(t+1) = \quad & \widetilde{d}_l(t) + \frac{1}{C_l} \sum_{j: l \in R_j} \widetilde{x}_j(t),\ t = 0,1,\ldots;\ l \in \mathcal{L}, \\ & t = 0,1,\ldots;\ l \in \mathcal{L},\ i \in \mathcal{M}. \end{aligned} \tag{14}$$

which can be described in matrix form as

$$\begin{pmatrix} \tilde{\mathbf{x}}(t+1) \\ \widetilde{d}(t+1) \end{pmatrix} = \mathbf{G} \begin{pmatrix} \tilde{\mathbf{x}}(t) \\ \widetilde{d}(t) \end{pmatrix}. \tag{15}$$

For the system (14) to be locally stable around the equilibrium point the eigenvalues λ of the matrix $\mathbf{G}$ have to be in the open unit circle, or $|\lambda| < 1$. We next study this condition only in the α-β parameter space defined by the vector $p = [\alpha_1, \ldots, \alpha_M, u_1, \ldots, u_M, C_1, \ldots, C_L]^2$. In addition, connections between users as described by the routing matrix $\mathbf{A}$ can also be taken as a variable, extending the parameter space to that described by the extended vector

$$\begin{aligned} p = \quad & [\alpha_1, \ldots, \alpha_M, u_1, \ldots, u_M, C_1, \ldots, C_L, \\ & \{A_{l,i}, l \in \mathcal{L}, i \in \mathcal{M}\}], \end{aligned}$$

where we study stability of the network under all possible routing configurations for a given number of users and nodes.

1) Illustrative Example: We first study at the effect of capacity on stability of the linearized system (15) using an illustrative example with 3 users and 2 links. The numbers of users and links are chosen small in order to be able to visualize the results. The routing matrix is fixed in this example, and is given by $\mathbf{A} \equiv \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$. The corresponding network configuration is shown in Fig. 3. The matrix $\mathbf{G}$ for this example can now be written out explicitly as

$$\begin{pmatrix} 1-\beta_1 & 0 & 0 & -\alpha_1 & -\alpha_1 \\ 0 & 1-\beta_2 & 0 & -\alpha_2 & 0 \\ 0 & 0 & 1-\beta_3 & 0 & -\alpha_3 \\ \frac{1}{C_1} & \frac{1}{C_1} & 0 & 1 & 0 \\ \frac{1}{C_2} & 0 & \frac{1}{C_2} & 0 & 1 \end{pmatrix}.$$

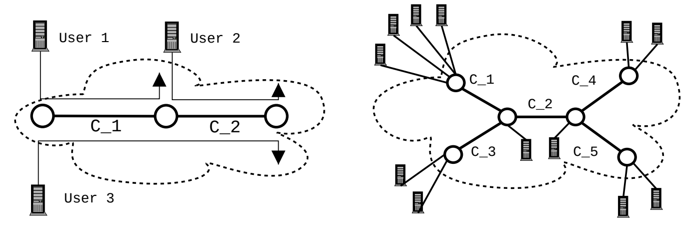

Fig. 3. Network diagram for the illustrative example (a) and the general network topology with 12 users and 5 links.

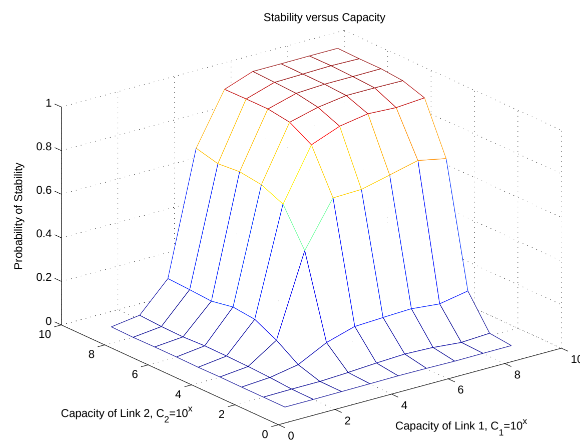

Fig. 4. Probability of stability for various capacities of links for the network in the illustrative example.

As in previous simulations, the parameter ranges are chosen as $0 < \beta_i < 1$ and $0 < \alpha_i < 1,000$, and a uniform distribution within the given range is used as the probabilistic model for the parameters. Capacities of the links are varied exponentially from 10^2 to 10^6. In Fig. 4 it is shown that probability of stability increases with increasing capacity of the links, which is consistent with earlier results on the single bottleneck link case.

2) Simulations under general network topologies: We next simulate the system under the arbitrary network topology of Fig. 3 with 5 links and 12 users. Note that, one can repeat this simulation for arbitrarily large networks. We investigate the local stability of the system for different parameter ranges varying from 0.1 to 1 for β and from 100 to 1,000 for α. A uniform distribution is used as probabilistic model within each given range of parameters. $C_1, \ldots, C_5$ are arbitrarily fixed to values $[35\ 50\ 30\ 15\ 20]\ 10^3$, and $N = 10,000$. As we observe

[2]The reason why we do not consider the $u - \alpha$ space is because the entries of the matrix in that case depend also on the equilibrium state, which however cannot be expressed in closed form in terms of the system parameters, since it is the solution of a set of nonlinear equations.

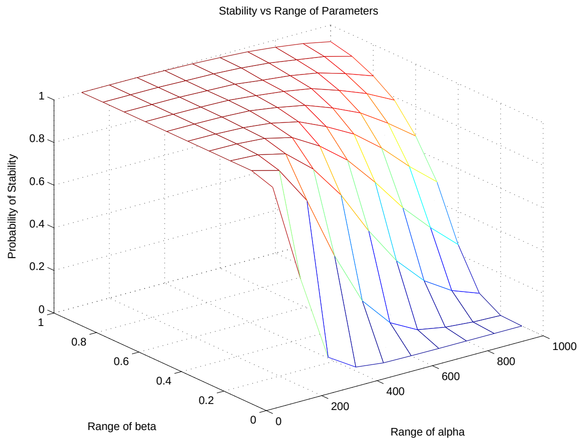

Fig. 5. Probability of stability for various ranges of parameters under the arbitrary network topology given by Figure 3.

from Figure 5, results are comparable to those obtained for the single bottleneck link case.

Finally, we investigate robustness of the linearized system to different routes. The number of users and links are the same as in the previous simulation. However, the routing matrix, and hence, the topology and routes in the network, are generated randomly in addition to the parameters which are generated using a uniform distribution within the ranges $0 < \beta_i < 1$ and $0 < \alpha_i < 1000\,,\ i = 1, \ldots, 4$. The number of different topologies randomly generated is 50, and number of samples per topology is chosen as $5,000$. We observe that the system is stable with a probability of 0.70. If we increase the capacities of the links tenfold to $[35\ 50\ 30\ 15\ 20]\,10^4$, however, the probability of stability increases to 0.99. As noted earlier, any simulation in the u-α space is not feasible under general network topologies with multiple bottleneck links, as the explicit calculation of the unique equilibrium state requires the solution of a set of nonlinear equations.

VI. Conclusion

In this paper, we have investigated the local stability and robustness of a discrete-time nonlinear congestion control algorithm, first at a single bottleneck link and then under general network topologies. For symmetric users at a single bottleneck link, we have obtained necessary and sufficient conditions for the local stability of the system. For more general scenarios, which include multiple bottleneck links and nonsymmetric users, analytic derivation is not possible. Hence, we have resorted to randomized algorithms, and made use of both Monte Carlo and Quasi-Monte Carlo methods. Specifically, we have used Halton, Niederreiter, and Sobol sequences as quasi-random sequences in addition to uniform random distributions created using standard pseudo-random number generators. As the quasi-random number generators technical restrictions in higher dimensions, we have used Monte Carlo methods for the analysis of systems with higher number of users.

This study reveals that randomized algorithms provide extensive insight into local stability of congestion control algorithms which are inherently nonlinear. Furthermore, one can obtain accurate results on stability margins even with a small number of samples. One reason for this may be that the linearized system has relatively simple stability boundaries in the parameter space. We conclude through simulations that under a given (general) network topology the nonlinear system is locally stable only for a specific range of values for the parameters, β, u, α, and C. Future work in this area includes application of randomized algorithms to stability and robustness analysis of various congestion control schemes, and possible extensions to wireless networks.

References

[1] T. Alpcan and T. Başar, "A Utility-Based Congestion Control Scheme for Internet-Style Networks with Delay," in *Proc. IEEE Infocom*, San Francisco, CA, March 30-April 3, 2003.

[2] V. Anantharam and J. Walrand (Guest Editors), "A Special Issue on Control Methods for Communication Networks," *Automatica*, vol. 35, December 1999.

[3] L. Bushnell (Guest Editor), "Special Selection on Networks and Control," *Control Systems Magazine*, vol. 21, February 2001.

[4] W. Gong and T. Başar (Guest Editors), "Special Issue of the IEEE Transactions on Automatic Control on Systems and Control Methods for Communication Networks," *IEEE Transactions on Automatic Control*, vol. 47, June 2002.

[5] G. Calafiore, F. Dabbene and R. Tempo, "Randomized Algorithms for Probabilistic Robustness with Real and Complex Structured Uncertainty," *IEEE Transactions on Automatic Control*, vol. AC-45, pp. 2218-2235, 2000.

[6] R. Tempo and F. Dabbene, "Randomized Algorithms for Analysis and Control of Uncertain Systems: An Overview," in *Perspectives in Robust Control*, Springer-Verlag, London, 2001.

[7] M. Vidyasagar, "Statistical Learning Theory and Randomized Algorithms for Control," *IEEE Control Systems Magazine*, vol. 18, pp. 69-85, December 1998.

[8] M. Vidyasagar, *A Theory of Learning and Generalization with Applications to Neural Networks and Control Systems*, Springer-Verlag, Berlin, 1996.

[9] L. Devroye, *Non-Uniform Random Variate Generation*, Springer-Verlag, New York, 1996.

[10] D. E. Knuth, *The Art of Computer Programming, Seminumerical Algorithms*, Addison-Wesley, Reading, MA, 1996.

[11] P. Hellekalek, "Good Random Number Generators are (not so) Easy to Find", *Mathematics and Computers in Simulation*, vol. 46, pp. 485-505, 1998.

[12] T. Başar and G. J. Olsder, *Dynamic Noncooperative Game Theory*, 2nd ed. Philadelphia, PA: SIAM, 1999.

[13] F. Kelly, A. Maulloo, and D. Tan, "Rate control in communication networks: Shadow prices, proportional fairness and stability," *Journal of the Operational Research Society*, vol. 49, pp. 237–252, 1998.

[14] J. E. Gentle, *Statistics and Computing: Random Number Generation and Monte Carlo Methods*, Springer, Berlin, 1996.

[15] H. Niederreiter, *Random Number Generation and Quasi-Monte Carlo Methods*, SIAM, Philadelphia, Pennsylvania, 1992.

[16] M. S. Branicky, S. M. LaValle, K. Olson, and L. Yang, "Deterministic vs. Probabilistic Roadmaps," submitted to the IEEE Transactions on Robotics and Automation, January 2002.