Randomized Algorithms for Stability and Robustness Analysis of High-Speed Communication Networks

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***Abstract*—This paper initiates a study toward developing and applying randomized algorithms for stability of high-speed communication networks. The focus is on congestion and delay-based flow controllers for sources, which are "utility maximizers" for individual users. First, we introduce a nonlinear algorithm for such source flow controllers, which uses as feedback aggregate congestion and delay information from bottleneck nodes of the network, and depends on a number of parameters, among which are link capacities, user preference for utility, and pricing. We then linearize this nonlinear model around its unique equilibrium point and perform a robustness analysis for a special symmetric case with a single bottleneck node. The "symmetry" here captures the scenario when certain utility and pricing parameters are the same across all active users, for which we derive closed-form necessary and sufficient conditions for stability and robustness under parameter variations. In addition, the ranges of values for the utility and pricing parameters for which stability is guaranteed are computed exactly. These results also admit counterparts for the case when the pricing parameters vary across users, but the utility parameter values are still the same. In the general nonsymmetric case, when closed-form derivation is not possible, we construct specific randomized algorithms which provide a probabilistic estimate of the local stability of the network. In particular, we use Monte Carlo as well as quasi-Monte Carlo techniques for the linearized model. The results obtained provide a complete analysis of congestion control algorithms for internet style networks with a single bottleneck node as well as for networks with general random topologies.**

***Index Terms*—Communication systems, congestion control, distributed control, Monte Carlo methods, pricing, robustness.**

I. Introduction

HIGH-speed communication networks recently received growing attention in the control literature, as evidenced by the appearance of several special issues devoted to this topic in leading journals in the field, such as [1], [2], and [3]. Various approaches and solutions have been developed and studied in this context, including modeling of TCP/IP traffic, congestion control for available bit rate (ABR) service in asynchronous transmission mode (ATM) networks, packet marking schemes for the Internet, application of low-order controllers for active queue management (AQM) as well as related problems.

One of the critical issues that lie at the heart of efficient operation of high-speed networks is *congestion control*. This involves the problem of regulating the source rates in a decentralized and distributed fashion, so that the available bandwidths on different links are used most efficiently while minimizing (or totally eliminating) loss of packets due to queues at buffers exceeding their capacities. All this has to be accomplished under variations in network conditions such as packet delays (due to propagation as well as queueing) and bottleneck nodes. This paper addresses this challenge, using *randomized algorithms*, within the context of a model introduced in [4], which is based on noncooperative game theory [5], and captures all the elements and features mentioned previously.

The congestion control problem and modeling of the Internet has been a very active research area in recent years. Modeling and analysis of congestion control algorithms have been the focus of this research after the introduction of the transfer control protocol (TCP) [6] and the mathematical models in [7] and [8], which pose the underlying resource allocation in congestion control as an optimization problem. Subsequent studies have proposed and analyzed various primal, dual, and primal-dual algorithms [9]–[14] building on this foundation. For a comprehensive summary of the results in this area, we refer to [15].

The original model utilized in this paper can be classified as a primal-dual algorithm [16]. It is nonlinear and in continuous time (CT), but we work here with a discrete-time (DT) version of it as any implementation of the CT model will inevitably involve a discretization synchronized with the round trip time (RTT) of packets. The DT model is also nonlinear, and depends on a number of parameters representing pricing, utility (to individual users), and link capacities. It has a unique equilibrium state for each set of values of these parameters, and the objective is to establish stability in a region of the parameter space taken as large as possible. The presence of the nonlinearities in the DT model makes it impossible to obtain a global stability result (even though this is possible for the CT version [4]), which forces us to study the linearized version around the equilibrium state, which is also of independent interest. Although there exist a small number of studies in the literature that investigate stability of linear and nonlinear primal or dual DT models, the analysis of the primal-dual DT model we focus on is novel to the best of our knowledge. The goal, then restated, is to establish local stability and robustness (to parameter variations) of this DT model.

However, even this goal cannot be accomplished analytically, due to nonlinear dependence of the linearized model on the key

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system parameters, except in some special cases. One such scenario is the so-called *user-symmetric* case, which corresponds to the situation when certain utility network parameters are all equal. For the more general case one has to resort to other (non-analytic) tools, among which the approach using *randomized algorithms* [17], [18] stands out as a strong contender.

The study of randomized algorithms for analysis and design of control systems has aroused considerable interest in the systems and control community [17]. They are efficient and low-complexity algorithms, and are useful especially when worst case analysis of complex systems is either very difficult or impossible. Unlike more classical methods, these algorithms yield an assessment on the satisfaction of required specifications with a certain probabilistic accuracy. They provide an alternative solution with a tradeoff between computational complexity and tightness of the solution.

Randomized algorithms heavily rely on univariate and multivariate methods for sample generations in various sets [19]. Roughly speaking, sample generation techniques can be divided into two main categories: Monte Carlo [20] and quasi-Monte Carlo [21]. While the former is classical, statistical-based, and assumes an *a priori* knowledge of probability density functions, the latter may be regarded as its deterministic counterpart. The main objective of quasi-Monte Carlo methods is to reduce the "discrepancy" between the generated samples, and a secondary objective is to avoid the curse of dimensionality that arises in gridding or rejection methods. Specific comparisons between these methods have already been performed in different areas, including the computation of path integrals in mathematical finance [22], and the analysis of various motion planning problems [23]. Even though quasi-Monte Carlo methods may be more suitable in certain situations, a definitive conclusion is not yet available. Therefore, one of the contributions of this paper is to provide additional insight in this direction for networking problems-an unexplored domain for randomized algorithms. The results obtained turn out to be very appealing from a networking point of view, as they carve out a sufficiently large region in the parameter space where local stability is ensured, implying that both flow rates of individual users and delays on links leading to bottleneck nodes stay around their equilibrium values which also admit the interpretation as Nash equilibrium when the number of users is sufficiently large.

The paper is organized as follows. In Section II, we present the general network model and the congestion control algorithm. An analytical local stability analysis of the single bottleneck node case with symmetric users is given in Section III. In Section IV, we give a brief account of randomized algorithms and discuss their use in the present context. In Section V, numerical results are presented for stability of the linearized system first for a single bottleneck node and subsequently under general network topologies. Conclusions are then provided in Section VI.

II. Model

A. Network Model

We consider a somewhat simplified version of a general network model based on fluid approximations introduced in [4]. Fluid models which replace discrete packets with continuous flows are widely used in addressing a variety of network control problems such as congestion control [9], [24], [25], routing [24], [26], and pricing [8], [27], [28]. The topology of the network studied here is characterized by a set of nodes $\mathcal{N}$ and a set of links $\mathcal{L}$, with each link $l \in \mathcal{L}$ having a fixed capacity $C_l > 0$, and an associated buffer size $b_l \geq 0$. There are M users, with the users set denoted by $\mathcal{M} := \{1, \dots, M\}$. Each user is associated with a unique *connection* between a source and a destination node. The connection is a path that connects various nodes, which can also be viewed as a subset of $\mathcal{L}$. The nonnegative flow, x_i, sent by the ith user over this path satisfies the bounds $0 \leq x_i \leq x_{i_{\max}}$. The upper bound, $x_{i_{\max}}$, on the ith user's flow rate may be a user specific physical limitation.

It is possible to define a routing matrix, $\mathbf{A}$, as in [8], that describes the relation between the set of routes $\mathcal{R}$ associated with the users (connections) and links

$$A_{l,i} = \begin{cases} 1, & \text{if source i uses link } l \\ 0, & \text{if source i does not use link } l \end{cases} \quad (1)$$

for $i \in \mathcal{M}$ and $l \in \mathcal{L}$. Using the routing matrix $\mathbf{A}$, we have the inequality

$$\mathbf{A}\mathbf{x} \leq \mathbf{c} \quad (2)$$

where $\mathbf{x}$ is the $(M \times 1)$ flow rate vector of the users and $\mathbf{c}$ is the $(L \times 1)$ link capacity vector. If the aggregate sending rate of users whose flows pass through link l exceeds the capacity C_l of the link, then the arriving packets are queued (generally on a first-come first-serve basis) in the buffer b_l of the link. Let the total flow on link l at any time t be given by

$$\bar{x}_l(t) := \sum_{i:l \in R_i} x_i(t). \quad (3)$$

Ignoring boundary effects, the buffer level at link l evolves according to

$$\dot{b}_l(t) = \bar{x}_l(t) - C_l \quad (4)$$

where $\dot{b}_l(t)$ denotes the partial derivative $\partial b_l(t)/\partial t$.

B. Cost Function

An important indication of congestion for Internet-style networks is the variation in queueing delay d, defined as the difference between the actual delay d^a experienced by a packet and the propagation delay d^p of the connection. If the incoming flow rate to a router exceeds the capacity of the outgoing link, packets are queued (generally on a first-come first-serve basis) in the corresponding buffer of the router, leading to an increase in the round-trip time (RTT) of packets. Hence, RTT on a congested path is longer than the base RTT, which is defined as the sum of propagation and processing delays on the path of a packet. The queueing delay at a link can be modeled as

$$\dot{d}_l(\mathbf{x}, t) := \frac{\partial d_l}{\partial t} = \frac{1}{C_l}(\bar{x}_l(t) - C_l) \quad (5)$$

where C_l the capacity of link l, and $\sum_{i:l \in R_i} x_i$ the total flow on the link. Thus, the queueing delay that a user experiences is the sum of queueing delays on its path, that is $D_i(\mathbf{x},t) = \sum_{l\in R_i} d_l(\mathbf{x},t)$.

Remark II.1: We ignore the positive projections in (4) and (5) due to the fact that we focus on local analysis and the point of interest is inside the boundaries.

We make use of variations in RTT to devise a congestion control and pricing scheme. The cost function for the ith user at time t is the difference between a linear pricing function proportional to the queueing delay the user experiences and a strictly increasing logarithmic utility function multiplied by a user preference parameter u_i

$$J_i(\mathbf{x},t) = \alpha_i\, D_i(\mathbf{x},t)\, x_i \, - u_i \log(x_i+1). \quad (6)$$

The utility function models the user's demand for bandwidth. It is strictly increasing and concave in accordance with the principle of diminishing returns. The pricing function is proportional to variations in the RTT a user experiences. A similar approach has been suggested in a version of TCP, known as TCP Vegas [29] as an *ad hoc* improvement over TCP Reno [30].

The users pick their flow rates in a way that would minimize their cost functions (and in this sense we have a noncooperative game), and consistent with this goal we adopt a dynamic update model whereby each user changes his flow rate proportional to the gradient of his cost function with respect to his flow rate. Thus, the algorithm for the ith user is

$$\frac{dx_i}{dt} = \dot{x}_i = -\frac{\partial J_i(\mathbf{x})}{\partial x_i} = \frac{u_i}{x_i+1} - \alpha_i D_i \quad (7)$$

where we have ignored the effect of the ith user's flow on the delay D_i that s/he experiences. This assumption can be justified for networks with a large number of users.

In a realistic implementation of the algorithm, the users update their flow rates only at discrete time instances corresponding to multiples of RTT and, hence, we discretize the reaction function of the ith user, and normalize it with respect to the RTT of the user. The optimal user response is, therefore, a DT version of (7), and is given by[1]

$$x_i(t+1) = x_i(t) + \kappa_i \left[\frac{u_i}{x_i(t)+1} - \alpha_i \sum_{l\in R_i} d_l(t)\right] \quad t = 0,1,\ldots, \quad i \in \mathcal{M} \quad (8)$$

where κ_i is a (user specific) step-size constant which will be taken to be 1 for the rest of the paper, which is no loss of generality since it can be absorbed into the other parameters u_i and α_i. Furthermore we take $x_i(0) = 0\,, i \in \mathcal{M}$, without any loss of generality. The queue model is discretized in a similar manner, with the queueing delay at link l being [as a discretized version of (5)]

$$d_l(t+1) = d_l(t) + \frac{1}{C_l}\sum_{i:l\in R_i} x_i(t) - 1, \quad t = 0,1,\ldots \quad (9)$$

with $d_l(0) = 0, l \in \mathcal{L}$.

[1]We have abused the notation here, as t here does not correspond to the t in the CT description. Since the CT description will not be used in the rest of the paper, this should not create any ambiguity or confusion.

III. Stability Analysis: The Symmetric Single Bottleneck Case

Let us consider the case of a single bottleneck node, with M users having connections passing through that node. Hence, we have essentially a single link of interest, for which we denote the associated delay by d (that is without the subscript 'l'), and likewise the associated capacity by C. Then, the equilibrium state of the system described by (8) and (9) follows readily as:

$$x_i^* = \frac{u_i}{\alpha_i d^*} - 1, \quad i \in \mathcal{M}$$
$$d^* = \frac{1}{C+M}\sum_{i=1}^{M}\frac{u_i}{\alpha_i} \quad (10)$$

which is unique.

Let $\tilde{x}_i(t) := x_i(t) - x_i^*, i \in \mathcal{M}$, and $\tilde{d}(t) := d(t) - d^*$. The system (8)–(9) with a single bottleneck link and with $\kappa_i = 1$ can now be rewritten around the equilibrium state as

$$\tilde{x}_i(t+1) = \tilde{x}_i(t) + \frac{u_i}{\tilde{x}_i(t) + x_i^* + 1} - \alpha_i(\tilde{d}(t) + d^*) \quad i \in \mathcal{M}$$
$$\tilde{d}(t+1) = \tilde{d}(t) + \frac{1}{C}\sum_{i=1}^{M}\tilde{x}_i(t). \quad (11)$$

Let $\tilde{\mathbf{x}}$ denote the M-dimensional column vector whose entries are the $\tilde{x}_i$'s. Likewise let the M-dimensional column vector whose entries are the $\tilde{x}_i^*$'s be denoted by $\tilde{\mathbf{x}}^*$. Linearizing (11) around $(\tilde{\mathbf{x}}^*, \tilde{d}^*) = (0,0)$, we obtain

$$\tilde{x}_i(t+1) = \left[1 - \frac{u_i}{(x_i^*+1)^2}\right]\tilde{x}_i(t) - \alpha_i \tilde{d}(t), \quad i \in \mathcal{M}$$
$$\tilde{d}(t+1) = \tilde{d}(t) + \frac{1}{C}\sum_{i=1}^{M}\tilde{x}_i(t). \quad (12)$$

Let

$$\alpha := [\alpha_1, \alpha_2, \ldots, \alpha_M] \quad \text{and} \quad \beta := [\beta_1, \beta_2, \ldots, \beta_M]$$
$$\text{where } \beta_i := u_i/(x_i^*+1)^2, \quad i \in \mathcal{M}. \quad (13)$$

The system equations (12) can then be expressed in matrix form

$$\begin{pmatrix} \tilde{\mathbf{x}}(t+1) \\ \tilde{d}(t+1) \end{pmatrix} = \mathbf{L}(\alpha,\beta,C)\begin{pmatrix} \tilde{\mathbf{x}}(t) \\ \tilde{d}(t) \end{pmatrix} \quad (14)$$

where

$$\mathbf{L}(\alpha,\beta,C) = \begin{pmatrix} 1-\beta_1 & 0 & \cdots & -\alpha_1 \\ 0 & 1-\beta_2 & & -\alpha_2 \\ \vdots & & \ddots & \vdots \\ \frac{1}{C} & \frac{1}{C} & \cdots & 1 \end{pmatrix}. \quad (15)$$

Hence, the system (11) is locally asymptotically stable if and only if $\mathbf{L} = \mathbf{L}(\alpha,\beta,C)$ is *Schur*, that is all its eigenvalues, $\lambda(\alpha,\beta,C)$, are in the open unit circle. The goal now is to find the region in the parameter space (with the parameters being α, β, and C), such that $|\lambda(\alpha,\beta,C)| < 1$. We consider first the special case of symmetric users having the same pricing and utility preference parameters, that is $u_i = u$ and $\alpha_i = \alpha$ for all

$i \in \mathcal{M}$, which also implies that $\beta_i = \beta$ for all $i \in \mathcal{M}$. When the number of users is two, $M = 2$, one can explicitly determine the eigenvalues of the matrix $\mathbf{L}$. They are given by

$$\lambda_1 = 1 - \beta$$
$$\lambda_{2,3} = 1 - \frac{\beta}{2} \pm \sqrt{\frac{\beta^2}{4} - 2\frac{\alpha}{C}} \tag{16}$$

which are in the open unit circle if and only if

$$\frac{2\alpha}{C} < \beta < 2.$$

The lemma below and the proposition that follows generalizes this result to M users.

Lemma III.1: If the user preference parameters and prices are symmetric across all M users, that is $u_i = u$ and $\alpha_i = \alpha$ (which further implies that $\beta_i = \beta$), then the characteristic equation of the matrix $\mathbf{L}$ is given by

$$\det|\lambda I - \mathbf{L}| = (\lambda - (1-\beta))^{M-1} \times \left[\lambda^2 - (2-\beta)\lambda + 1 - \beta + \frac{M\alpha}{C}\right]. \tag{17}$$

Thus, $\mathbf{L}$ has $M-1$ real eigenvalues at $1-\beta$ and two possibly complex eigenvalues at

$$1 - \beta/2 \pm \sqrt{\beta^2/4 - M\alpha/C}.$$

Proof: The lemma is proven by induction. It is already shown in (16) that the statement holds when $M = 2$. Next, we assume that the statement holds for a given M, say $M = m$, and prove that it also holds for $m+1$. Now note that

$$\begin{aligned}\det|\lambda I_{m+2} - \mathbf{L}_{m+1}| &= (\lambda - (1-\beta)) \\ &\quad \times \left[(\lambda - (1-\beta))^{m-1}\left(\lambda^2 - (2-\beta)\lambda + 1 - \beta + \frac{m\alpha}{C}\right)\right] \\ &\quad + \frac{\alpha}{C}(\lambda - (1-\beta))^m \\ &= (\lambda - (1-\beta))^m \left[\lambda^2 - (2-\beta)\lambda + 1 - \beta + (m+1)\frac{\alpha}{C}\right].\end{aligned}$$

Thus, the given expression for the characteristic equation holds for $M = m+1$ and, hence, for all $M \geq 2$. □

We now determine the region in the parameter space where $\mathbf{L}$ is Schur matrix. It readily follows from the lemma that the condition $0 < \beta < 2$ is both necessary and sufficient for $M-1$ real roots $(1-\beta)$ to be in the open unit circle. On the other hand, the remaining two possibly complex roots of (17) have their absolute values strictly less than one, $|\lambda| < 1$, if and only if the following holds:

$$\beta \in \left(\min\left\{\frac{M\alpha}{C}, 2\sqrt{\frac{M\alpha}{C}}\right\}, 2 + \frac{M\alpha}{2C}\right).$$

Combining this with the earlier condition $0 < \beta < 2$, we arrive at the following necessary and sufficient condition for local stability of the equilibrium state of system (11) in the symmetric user case

$$\beta \in \left(\min\left\{\frac{M\alpha}{C}, 2\sqrt{\frac{M\alpha}{C}}\right\}, 2\right).$$

This, in turn, is equivalent to the condition

$$\frac{M\alpha}{C} < \beta < 2. \tag{18}$$

We summarize this result in the following proposition.

Proposition III.2: If the user preference parameters and prices are symmetric across all M users (that is, $u_i = u$ and $\alpha_i = \alpha$, which further implies that $\beta_i = \beta$), the single bottleneck system given by (11) is locally stable around its equilibrium state (10) if and only if the parameters α, β, and C lie in the region

$$\frac{M\alpha}{C} < \beta < 2.$$

Remark III.3: If the capacity of the link is linearly proportional to M (that is, $C = \mu$ M, for some positive constant μ), then the necessary and sufficient condition becomes

$$\frac{\alpha}{\mu} < \beta < 2.$$

The condition in Proposition III.2 can also be expressed in terms of the user preference parameter u, together with the pricing parameter α and capacity C. First note the relationship

$$\beta = \left(\frac{M}{C+M}\right)^2 u$$

which readily follows from (10) and (13) by taking u_i, α_i, and β_i independent of the user index i. In view of this relationship between β and u, we immediately have the following corollary to Proposition III.2.

Corollary III.4: For the symmetric parameter case, the single bottleneck system given by (11) is locally stable around its equilibrium state (10) if and only if the parameters α, u, and C lie in the region

$$\left(1 + \frac{M}{C}\right)\left(1 + \frac{C}{M}\right)\alpha < u < 2\left(1 + \frac{C}{M}\right)^2.$$

Finally, we generalize Proposition III.2 by removing the symmetry in the pricing parameter α, while retaining the symmetry in β.

Proposition III.5: Let the parameter β be symmetric across all M users, $\beta_i = \beta$, while the pricing vector $\alpha = [a_1, \ldots, a_M]$ be general. Then, the characteristic equation of the matrix $\mathbf{L}$ is given by

$$\det|\lambda I - \mathbf{L}| = (\lambda - (1-\beta))^{M-1} \times \left[\lambda^2 - (2-\beta)\lambda + 1 - \beta + \sum_{i=1}^{M}\frac{\alpha_i}{C}\right]. \tag{19}$$

The matrix $\mathbf{L}$ has $M-1$ real eigenvalues at $1-\beta$ and two possibly complex eigenvalues at

$$1 - \frac{\beta}{2} \pm \sqrt{\frac{\beta^2}{4} - \sum_{i=1}^{M}\frac{\alpha_i}{C}}.$$

Furthermore, the single bottleneck link system given by (11) is locally stable around its equilibrium state (10) if and only if the parameters $\{\alpha_i, i \in \mathcal{M}\}, \beta$, and C lie in the region

$$\frac{1}{C}\sum_{i=1}^{M}\alpha_i < \beta < 2.$$

Proof: The proof follows from the those of Lemma III.1 and Proposition III.2, by simply replacing $M\alpha$ with $\sum_{i=1}^{M}\alpha_i$.□

IV. Randomized Algorithms and Stability Analysis for the Nonsymmetric Case

We have seen in the previous section that local stability and robustness can be studied analytically (because the eigenvalues of $\mathbf{L}$ can be computed explicitly) when the user utility preference parameters, u_i's, are the same for all users (or equivalently when the β_i's are the same). If this is not the case, however, then the eigenvalues of $\mathbf{L}$ cannot be expressed in closed form, making it very challenging (if not impossible) to deduce any reasonable stability and robustness results using analytical techniques. Then, one has to resort to numerical-based or simulation-based methods, and as mentioned earlier *randomized algorithms* stand out as the most promising. However, before trying out randomized algorithms on the problem at hand, we first provide, in this section, a general introduction to the topic for the uninitiated reader. This section also serves to introduce the conceptual framework and the terminology, which will be utilized in the next section.

A. Monte Carlo Methods

In Monte Carlo methods, the first step is to take the parameter vectors α and β to be random with given probability density functions f_α and f_β, having support sets $\mathcal{B}_\alpha$ and $\mathcal{B}_\beta$, respectively. We can take, for example, $\mathcal{B}_\alpha$ and $\mathcal{B}_\beta$ to be the hyper-rectangular sets

$$\mathcal{B}_\alpha = \{\alpha : \alpha_i \in [\alpha_i^-, \alpha_i^+], i = 1, 2, \ldots, M\}$$
$$\mathcal{B}_\beta = \{\beta : \beta_i \in [\beta_i^-, \beta_i^+], i = 1, 2, \ldots, M\}$$

and the density functions f_α and f_β to be uniform on these sets. That is, for $i = 1, 2, \ldots, M$

$$f_{\alpha_i} = \begin{cases} \dfrac{1}{\alpha_i^+ - \alpha_i^-} & \text{if } \alpha_i \in [\alpha_i^-, \alpha_i^+] \\ 0 & \text{otherwise} \end{cases} \tag{20}$$

and

$$f_{\beta_i} = \begin{cases} \dfrac{1}{\beta_i^+ - \beta_i^-} & \text{if } \beta_i \in [\beta_i^-, \beta_i^+] \\ 0 & \text{otherwise} \end{cases}. \tag{21}$$

Then, we generate N independent identically distributed (i.i.d.) vector samples from the set $\mathcal{B}_\alpha(\mathcal{B}_\beta)$ according to the density function f_α: $\alpha^1, \alpha^2, \ldots, \alpha^N (f_\beta$: $\beta^1, \beta^2, \ldots, \beta^N)$, respectively. Subsequently, using (15) we compute $\mathbf{L}(\alpha^i, \beta^i)$ for $i = 1, 2, \ldots, N$, where we have suppressed the dependence on C.

The next step is to construct the indicator function

$$\mathcal{I}(\alpha^i, \beta^i) := \begin{cases} 1 & \text{if } \mathbf{L}(\alpha^i, \beta^i) \text{ is Schur} \\ 0 & \text{otherwise.} \end{cases}$$

The estimated *probability of stability* is readily given by

$$\hat{p}_N = \frac{1}{N}\sum_{i=1}^{N}\mathcal{I}(\alpha^i, \beta^i) \tag{22}$$

which is equivalent to

$$\hat{p}_N = \frac{N_{\text{good}}}{N}$$

where N_{good} is the number of vector samples such that $\mathbf{L}(\alpha^i, \beta^i)$ is a Schur matrix. The estimate $\hat{p}_N$ is usually referred to as *empirical probability*.

Clearly, for a finite sample size, it is important to know how many samples N are needed to obtain a "reliable" probabilistic estimate $\hat{p}_N$. To this end, classical results, such as the Chernoff bound can be used. The Chernoff bound [31] states that for any $\epsilon \in (0, 1)$ and $\delta \in (0, 1)$ if

$$N \geq \frac{1}{2\epsilon^2}\ln\left(\frac{2}{\delta}\right) \tag{23}$$

then, with probability greater than $1-\delta$, we have $|\hat{p}_N - p_{\text{true}}| < \epsilon$, where p_{true} denotes the real probability of stability. It is important to remark, however, that the number of required vector samples is independent of the problem dimension, e.g., of the size of the matrix $\mathbf{L}(\alpha, \beta)$ and of the number of users, M. Hence, this problem independent explicit bound which can be computed *a priori* [17], [18].

As pointed out in Section I, an important issue in Monte Carlo methods is the development of efficient algorithms for sample generation in various sets according to different distributions. In particular, the problem is how to efficiently generate N vector samples (α^i, β^i) according to the given densities f_α and f_β, and support sets $\mathcal{B}_\alpha$ and $\mathcal{B}_\beta$.

For univariate density functions, this specific problem is equivalent to the one of generating uniform random numbers in the interval [0, 1]. Good random number generators are required to provide uniform and independent samples and they should be also reproducible and fast. It is well-known that computer methods for random generation produce only pseudorandom sequences, which may show cyclicities and correlations. The problem of univariate random number generation constitutes a whole field of study in its own. As a starting point for the reader interested in further details, we refer to [31] and [32]. We note that, even though this is a well-established topic, current research is performed with the objective to produce extremely fast and reliable algorithms for various applications including, in particular, cryptography.

The case of multivariate distributions is definitely more difficult. For general distributions and support sets, rejection methods can be used [33]. There are two kinds of rejection methods: The first one is based on the concept of rejection from a "dominating density." The second class of methods may be used for uniform densities and it performs rejection from an overbouding bounding set. These two methods are obviously related and a critical issue in both is their efficiency since the rejection rate may be exponential. Alternatively, adaptive Monte Carlo methods based on Markov Chains or Metropolis-like algorithms may be utilized but the critical issue is the rate of convergence, which may be slow.

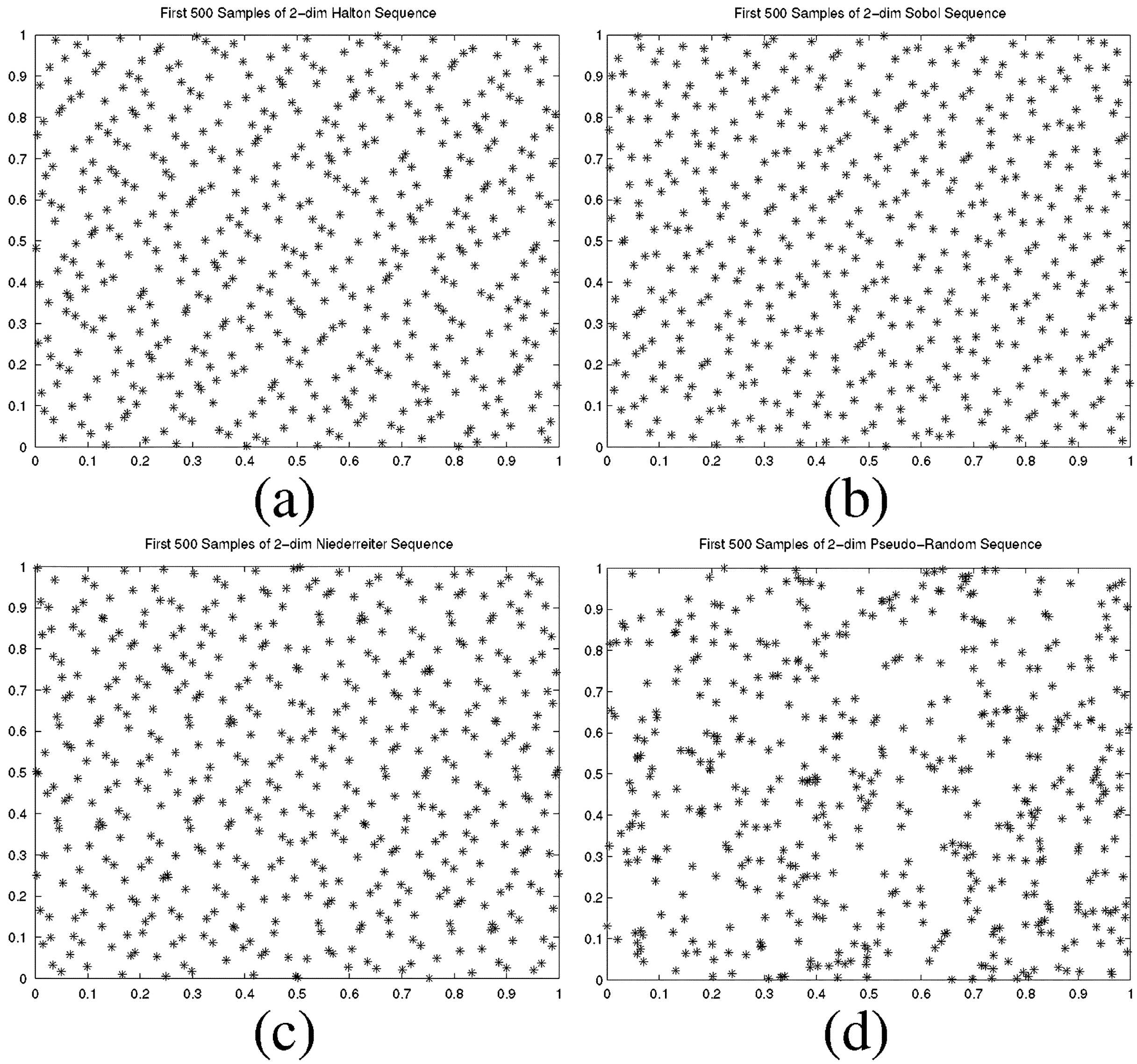

Fig. 1. First 500 samples of various two-dimensional quasi-random sequences. (a) Halton. (b) Sobol. (c) Niederreiter. (d) Uniformly distributed pseudorandom sequence.

B. Quasi-Monte Carlo Methods

In the case of quasi-Monte Carlo methods, the empirical probability can still be computed using the indicator function and (22) but the sample generation is obtained in a completely different way. That is, no probability density functions f_α and f_β are specified or used and the samples are generated according to a purely deterministic mechanism. Therefore, the sequences $\alpha^1, \alpha^2, \ldots, \alpha^N$ and $\beta^1, \beta^2, \ldots, \beta^N$ are now quasi-random and are chosen in order to minimize the so-called discrepancy, which is a measure of how "uniform" a sample set is distributed within a given set.

Formally, the discrepancy $D(S, \mathcal{B})$ of a sample set $S \in \mathcal{B}$ of cardinality N

$$S := s^1, s^2, \ldots, s^N$$

is defined as [21]

$$D(S, \mathcal{B}) = \sup_{B \in \mathcal{B}} \left| \frac{|S \cap B|}{N} - \mathrm{Vol}(B) \right| \tag{24}$$

where B is any subset of $\mathcal{B}$, $\mathrm{Vol}(B)$ is the volume of B and $|\cdot|$ denotes the cardinality of a set.

The idea is to "cover" the set $\mathcal{B}$ as uniformly as possible for a given sample size. One can ask, on the other hand, why a simple uniform grid providing low discrepancy is not preferred. Even though the apparent randomness of quasi-random sequences may be attractive for various reasons, the main benefit is to avoid the curse of dimensionality which is inherent to gridding techniques. That is, as the dimension of the parameter space increases, the number of samples required to cover the set $\mathcal{B}$ with a uniform grid grows exponentially. On the other hand, the advantage is that the number of samples in the quasi-Monte Carlo method is independent of the problem dimension, exactly as in the Monte Carlo method. Various classical low-discrepancy sequences are available in the literature, including Halton, Sobol, Niederreiter and others. Some plots showing specific generations are shown in Fig. 1. Finally, we would like to mention that discrepancy is not the only criterion used. For example, the so-called *dispersion*, which is a normalized lower bound on the discrepancy, is also studied. We do not further dwell on this issue, but we refer to [21] and [23] for additional details.

V. Simulation Results

In Section III, we have investigated the range of values for the parameters (α, β, C) for which the system is locally stable in the special symmetric parameter case. This analytical approach, however, cannot be further generalized, as it becomes extremely difficult to find a closed-form expression for the eigenvalues of the matrix $\mathbf{L}$. The uncertainty in the general case consists of nonlinearly coupled parameters even in the single bottleneck link case as shown in (15). Therefore, the use of randomized algorithms instead of classical worst case analysis is a natural choice for investigating the robustness of the system at hand. For the remainder of the paper, the term stability will be used in the probabilistic sense, referring to probability of stability, or its deterministic counterpart [17] if quasi-Monte Carlo is used.

In order to gain further insight into the properties of $\mathbf{L}$, the eigenvalues of a single 20-dimensional randomly generated sample $\mathbf{L}$ matrix are calculated and shown in Fig. 2. We note that $\mathbf{L}$ is ill-conditioned with a condition number in the order of 10^5.

We next investigate stability properties of the linearized single bottleneck link system (12) through simulations, and then generalize the simulations to cover multiple bottleneck systems. Our main goal is to investigate the effect of pricing and user parameters on local stability of the system.

A. Single Bottleneck Link With Multiple Users

We have seen earlier in Section III, that it is possible to study local stability and robustness in two different parameter spaces, namely (α, u, C) and (α, β, C), where the former admits an interpretation in terms of the original model, whereas the latter is just a transformation which was introduced for convenience. In any analytical study, such as the one in Section III, it does not make any difference whether one works with the former or the latter, since there is a one-to-one transformation between the two parameter spaces. In the case of randomized algorithms, however, it does make a difference, since the distribution one uses for one space does not necessarily correspond to the one used for the other. For this reason we carry out the analysis with randomized algorithms in both parameter spaces.

For the case of a single bottleneck link network, we first study local stability and robustness in the $u - \alpha$ parameter space

$$p = [\alpha_1, \ldots, \alpha_M, u_1, \ldots, u_M, C].$$

The matrix in question is (15), which is expressed in terms of β_i's, which however can be expressed in terms of u_i's through (13).

Subsequently, we carry out the study in the $\alpha - \beta$ parameter space, where the p vector is now

$$p = [\alpha_1, \ldots, \alpha_M, \beta_1, \ldots, \beta_M, C].$$

In both cases, we use not only Monte Carlo methods as the probabilistic model for the system, but also quasi-Monte Carlo sequences like, Halton, Sobol, and Niederreiter in order to-presumably-obtain a better coverage of the $2M + 1$-dimensional parameter space.

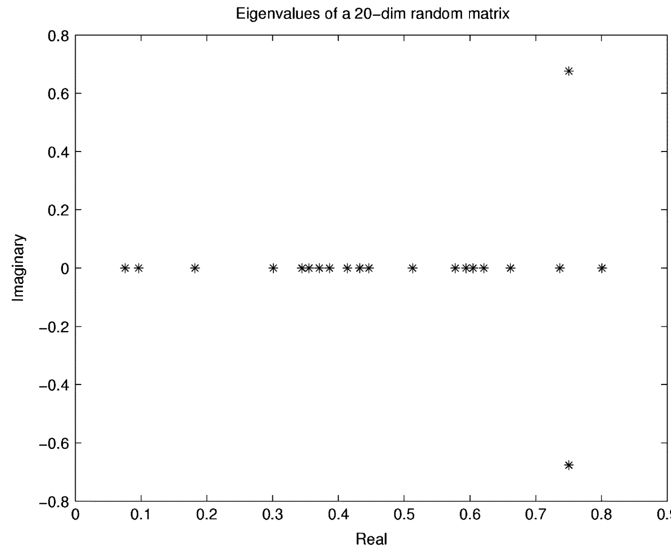

Fig. 2. Eigenvalues of a single 20-dimensional randomly generated sample $\mathbf{L}$ matrix on complex plane.

1) $u - \alpha$ Parameter Space: We first simulate the effect of bottleneck link capacity C on the local stability of the system for various values of u and α. In this simulation we use Monte Carlo, quasi-Monte Carlo and grid methods together, which enables us to compare the performance of these methods. Note that the grid method is the most reliable one as it covers the parameter space deterministically. However, it is prohibitive due to its computational complexity in higher dimensional systems. Due to this limitation, we simulate a system with four users only.

For all methods, the parameter ranges $0 < \alpha_i < 0.2$ and $0 < u_i < 20000, i = 1, \ldots, 4$, are chosen with 100% tolerance around their nominal values. For the probabilistic model for parameters, we use a uniform distribution. We choose a level of confidence $\delta = 0.001$ and accuracy $\varepsilon = 0.008$. Using the Chernoff bound given in (23) we determine the minimum sample size: $N \geq 59383$.To simplify the grid construction we choose $N = 65536$, which guarantees for the Monte Carlo simulation with probability greater than 0.999 that $|\hat{p}_N - p_{\mathrm{true}}| < 0.008$. Then, on the unit interval [0, 1], the grid is constructed through points spaced as $[0.125 \ 0.375 \ 0.625 \ 0.875]$ in each dimension. Results of this simulation are shown in Fig. 3. We observe that the system is locally stable only for a certain range of capacity C. Considering the analysis for the symmetric case given in Corollary III.4, this result aligns with the theoretical predictions.

We have observed a series of simulations that all of the implementations of quasi-Monte Carlo algorithms that we use have limitations as dimension of the system increases. The implementations of quasi-Monte Carlo algorithms that we use, for dimensions higher than 16, output sequences with a very specific pattern, which produces unreliable results. Hence, for a large number of users, we limit our analysis to Monte Carlo methods only.

Finally, we investigate the robustness of the system with respect to various user and pricing parameters given a fixed capacity at the bottleneck link. For each case, the user and pricing

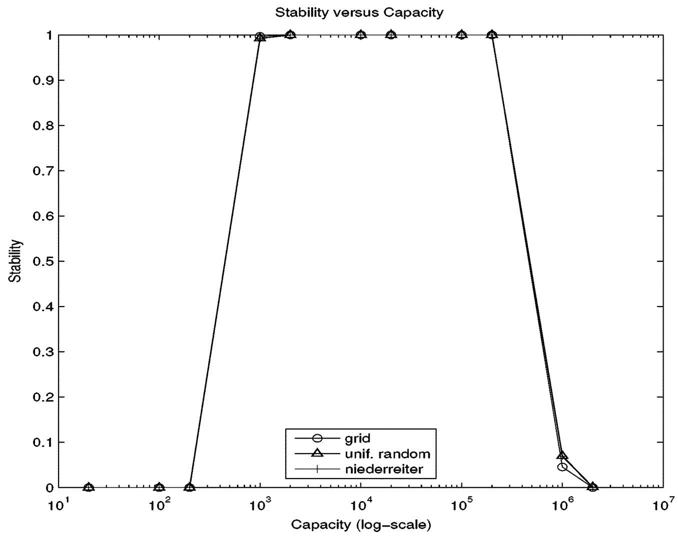

Fig. 3. Stability versus capacity (logarithmic scale) for $M = 4$ users using Monte Carlo, quasi-Monte Carlo, and grid methods. Parameters have 100% tolerance around nominal values.

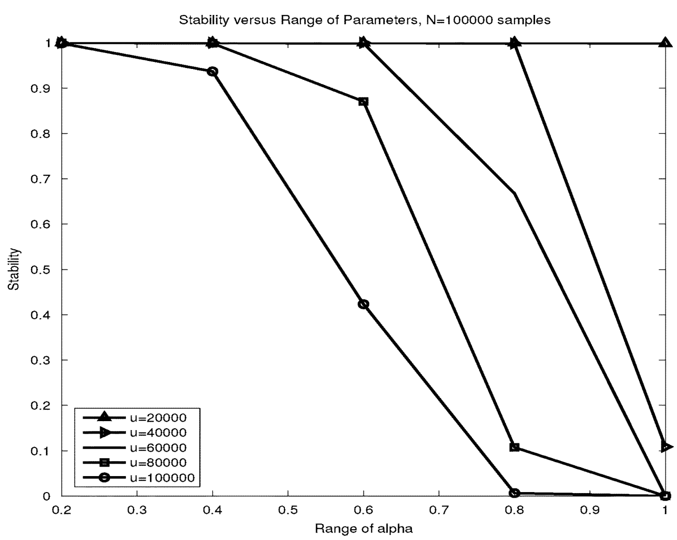

Fig. 4. Stability versus parameter ranges for $M = 20$ users. Parameters have 100% tolerance around nominal values.

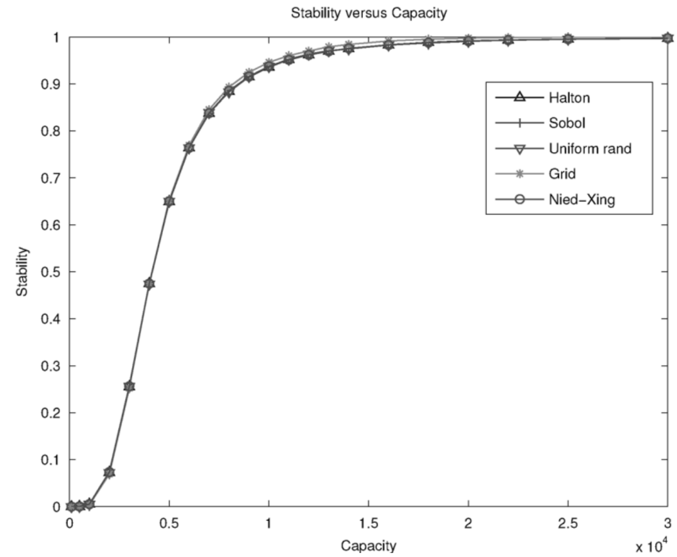

Fig. 5. Stability versus capacity for $M = 4$ users using Monte Carlo, quasi-Monte Carlo, and grid methods. Parameter ranges are $0 < \beta_i < 1$ and $0 < \alpha_i < 1000, i = 1, \ldots, 4$.

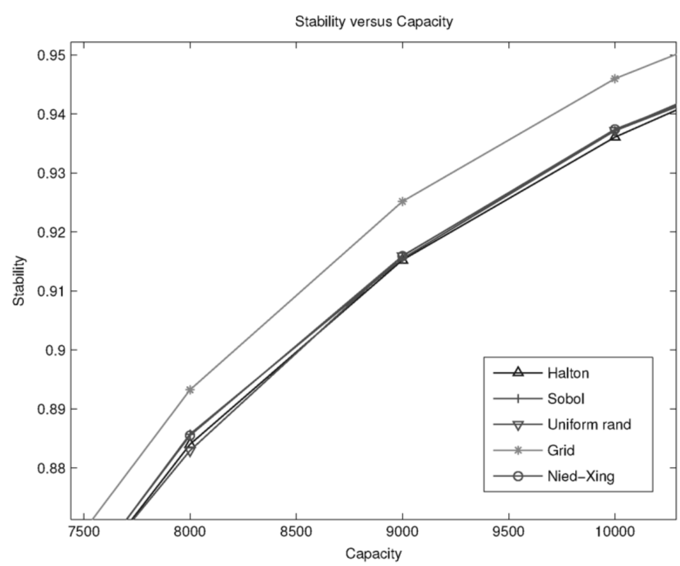

Fig. 6. Closer look at Fig. 5.

parameters are uniformly distributed with up to 100% tolerance around their nominal values. The capacity of the system is chosen as $C = 200000$, and number of users $M = 20$. We choose the number of samples as $N = 100000$, which easily guarantees a level of confidence $\delta = 0.001$ and accuracy $\varepsilon = 0.007$. We observe in Fig. 4 that local stability decreases as nominal values of u and α increase. As before, this observation is in line with the analytical results given in Corollary III.4.

2) $\alpha - \beta$ Parameter Space: We now carry out the preceding analysis in the $\alpha - \beta$ parameter space. As noted earlier, the $\alpha - \beta$ space is a nonlinear transformation of the $u - \alpha$ space, and, hence, any sample distribution in the former corresponds to some other sample distribution in the latter.

We first look at the effect of capacity. We simulate a system with again four users. For all the methods, the parameter ranges are taken to be $0 < \beta_i < 1$ and $0 < \alpha_i < 1000, i = 1, \ldots, 4$. As the probabilistic model for parameters, we use a uniform distribution. The grid is constructed through points spaced as $[0.1\ 0.3\ 0.5\ 0.7\ 0.9] \times R_{\max}$ in each dimension, where $R_{\max}$ is 1 for β and 1000 for α. The number of samples is then $N = 390\,625$, which guarantees a level of confidence $\delta = 0.001$ and accuracy $\varepsilon = 0.004$ for the Monte Carlo simulation. Results of this simulation are shown in Fig. 5, and a close-up version in Fig. 6. We observe that the stability of the system improves as capacity C increases as indicated by the condition (18) and analytical results in Propositions III.2 and III.5.

In the next simulation, we investigate robustness of the system with respect to a range of parameters $(\alpha - \beta)$ using uniform random distribution within the ranges [0, 1] for β and [0, 1000] for α. The capacity of the system is chosen as $C = 25000$, and number of users $M = 20$. The number of samples is $N = 100000$ ensuring a level of confidence $\delta = 0.001$ and accuracy $\varepsilon = 0.007$. We observe in Fig. 7 that

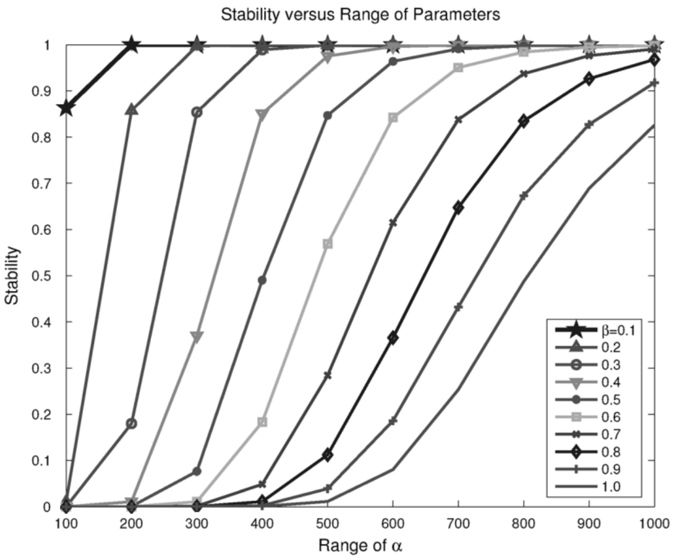

Fig. 7. Stability versus parameter ranges for $M = 20$ users using a uniform random distribution within the range.

local stability degrades as the ranges of β_i's and α_i's increase confirming analytical results in Propositions III.2 and III.5.

B. General Network Topology

We now turn our attention to local stability and robustness of a general topology network with multiple bottleneck links, and routing matrix $\mathbf{A}$ as described in (1). The system equations are given in (8) and (9). For this general case, equilibrium point or points of the system cannot be described explicitly. Therefore, we first investigate the uniqueness of the equilibrium state. Toward this end, we assume that $\mathbf{A}$ is a full row rank matrix with $M \geq L$ which is in fact no loss of generality as nonbottleneck links on the network have no effect on the equilibrium point, and can be safely left out. The following proposition can also be found in [4] and is included here for completeness.

Proposition V.1: When $\mathbf{A}$ is full row rank, the system described by (8) and (9) has a unique equilibrium.

Proof: The equilibrium state of the system described by (8) and (9) is

$$\mathbf{A}\mathbf{x} = \mathbf{C} \tag{25}$$

$$\mathbf{f}(\alpha, \mathbf{x}) = \mathbf{A}^T \mathbf{d} \tag{26}$$

where $\mathbf{d} = [d_1, \ldots, d_L]$ is the delay vector at links, and the nonlinear vector function $\mathbf{f}$ is defined as

$$\mathbf{f}(\alpha, \mathbf{x}) := \left[\frac{u_1}{\alpha_1}\frac{1}{(x_1+1)}, \ldots, \frac{u_M}{\alpha_M}\frac{1}{(x_M+1)}\right].$$

Multiplying (26) from left by $\mathbf{A}$ yields

$$\mathbf{A}\mathbf{f}(\alpha, \mathbf{x}^*) = \mathbf{A}\mathbf{A}^T\mathbf{d}.$$

Since $\mathbf{A}$ is of full row rank, the square matrix $\mathbf{A}\mathbf{A}^T$ is full rank and, hence, invertible. Thus, for a given flow vector $\mathbf{x}$ and pricing vector α

$$\mathbf{d}(\alpha, \mathbf{x}) = (\mathbf{A}\mathbf{A}^T)^{-1}\mathbf{A}\mathbf{f}(\alpha, \mathbf{x}) \tag{27}$$

is unique. Furthermore, we conclude that there is at least one equilibrium solution, $(\mathbf{x}^*, \mathbf{d}^*)$ which satisfies (25) and (26).

We next establish the uniqueness of the equilibrium. Suppose that there are two different equilibrium points, $(\mathbf{x}_1^*, \mathbf{d}_1^*)$ and $(\mathbf{x}_2^*, \mathbf{d}_2^*)$. Then, from (25) it follows that:

$$\mathbf{A}(\mathbf{x}_1^* - \mathbf{x}_2^*) = 0 \Leftrightarrow (\mathbf{x}_1^* - \mathbf{x}_2^*)^T\mathbf{A}^T = 0.$$

Similarly, from (26) we have

$$\mathbf{f}(\alpha, \mathbf{x}_1^*) - \mathbf{f}(\alpha, \mathbf{x}_2^*) = \mathbf{A}^T(\mathbf{d}_1^* - \mathbf{d}_2^*).$$

Multiplying this with $(\mathbf{x}_1^* - \mathbf{x}_2^*)^T$ from left we obtain

$$(\mathbf{x}_1^* - \mathbf{x}_2^*)^T[\mathbf{f}(\alpha, \mathbf{x}_1^*) - \mathbf{f}(\alpha, \mathbf{x}_2^*)] = 0.$$

We rewrite this as

$$\sum_{i=1}^{M}(\mathbf{x}_{1i}^* - \mathbf{x}_{2i}^*)\frac{1}{\alpha_i}\left[\frac{dU_i(x_{1i}^*)}{dx_i} - \frac{dU_i(x_{2i}^*)}{dx_i}\right] = 0.$$

Since U_i's are strictly concave, each term (say the ith one) in the summation is negative whenever $x_{1\,i}^* \neq x_{2\,i}^*$ with equality holding only if $x_{1i}^* = x_{2i}^*$. Hence, we conclude that x^* has to be unique, that is

$$\mathbf{x}^* = \mathbf{x}_1^* = \mathbf{x}_2^*.$$

From this, and (8)–(9), it immediately follows that $D_i, i = 1, \ldots, M$, are unique. This does not however immediately imply that $d_l, l = 1, \ldots, L$, are also unique, which in fact may not be the case if $\mathbf{A}$ is not full row rank. The uniqueness of d_l's, however, follow from (27), where we obtain a unique $\mathbf{d}^*$ for a given equilibrium flow vector $\mathbf{x}^*$

$$\mathbf{d}^* = (\mathbf{A}\mathbf{A}^T)^{-1}\mathbf{A}\mathbf{f}(\alpha, \mathbf{x}^*).$$

As a result, $(\mathbf{x}^*, \mathbf{d}^*)$, obtained from (25) and (26) constitutes a unique equilibrium point for the system (8)–(9). □

Let $\tilde{x}_i(t) = x_i(t) - x_i^*$ for the ith user and $\tilde{d}_l(t) = d_l(t) - d_l^*$ for the lth link, given the existence of a unique equilibrium point, $(\mathbf{x}^*, \mathbf{d}^*)$. The system (8)-(9), with $\kappa = 1$, can be rewritten around the equilibrium state as

$$\tilde{x}_i(t+1) = \tilde{x}_i(t) + \frac{u_i}{\tilde{x}_i(t) + x_i^* + 1} - \alpha_i \sum_{l \in R_i}(\tilde{d}_l(t) + d_l^*)$$
$$\tilde{d}_l(t+1) = \tilde{d}_l(t) + \frac{1}{C_l}\sum_{j:l \in R_j}\tilde{x}_j(t) \tag{28}$$

where $t = 0, 1, \ldots; l \in \mathcal{L}, i \in \mathcal{M}$. Linearizing the system (28) around the equilibrium point $(\tilde{\mathbf{x}}^*, \tilde{d}^*) = (0, 0)$, we obtain

$$\tilde{x}_i(t+1) = \left[1 - \frac{u_i}{(x_i^*+1)^2}\right]\tilde{x}_i(t) - \alpha_i \sum_{l \in R_i}\tilde{d}_l(t)$$
$$\tilde{d}_l(t+1) = \tilde{d}_l(t) + \frac{1}{C_l}\sum_{j:l \in R_j}\tilde{x}_j(t) \tag{29}$$

which can be described in matrix form as

$$\begin{pmatrix}\tilde{\mathbf{x}}(t+1)\\ \tilde{d}(t+1)\end{pmatrix} = \mathbf{G}\begin{pmatrix}\tilde{\mathbf{x}}(t)\\ \tilde{d}(t)\end{pmatrix}.$$

For the system (28) to be locally stable around the equilibrium, the eigenvalues λ of the matrix $\mathbf{G}$ have to be in the open unit circle, or $|\lambda| < 1$. We next study this condition only in the $\alpha - \beta$ parameter space. The reason why we do not consider the

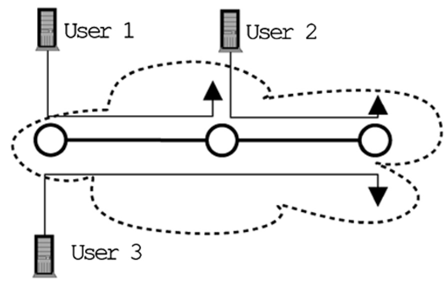

Fig. 8. Network diagram for Example V.B.2.

$u - \alpha$ space is because the entries of the matrix in that case depend also on the equilibrium state, which however cannot be expressed in closed form in terms of the system parameters, since it is the solution of a set of nonlinear equations.

1) $\alpha - \beta$ Parameter Space: We analyze the local stability and robustness of the linearized system (29) in $\alpha - \beta$ space, defined by the vector

$$p = [\alpha_1, \ldots, \alpha_M, \beta_1, \ldots, \beta_M, C_1, \ldots, C_L].$$

In addition, connections between users as described by the routing matrix $\mathbf{A}$ can also be taken as a variable, extending the parameter space to that described by the extended vector

$$p = [\alpha_1, \ldots, \alpha_M, \beta_1, \ldots, \beta_M, C_1, \ldots, C_L, \{A_{l,i}, l \in \mathcal{L}, i \in \mathcal{M}\}].$$

In this extended space, we study stability of the network under all possible routing configurations for a given number of users and nodes.

2) Illustrative Example: We first study at the effect of capacity on stability of the linearized system (V.B) using an illustrative example with three users and two links. The number of users and links is chosen small in order to be able to visualize the results. The routing matrix is fixed in this example, and is given by

$$\mathbf{A} \equiv \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

The corresponding network configuration is shown in Fig. 8. The matrix $\mathbf{G}$ for this example can now be written out explicitly as

$$\begin{pmatrix} 1-\beta_1 & 0 & 0 & -\alpha_1 & -\alpha_1 \\ 0 & 1-\beta_2 & 0 & -\alpha_2 & 0 \\ 0 & 0 & 1-\beta_3 & 0 & -\alpha_3 \\ \frac{1}{C_1} & \frac{1}{C_1} & 0 & 1 & 0 \\ \frac{1}{C_2} & 0 & \frac{1}{C_2} & 0 & 1 \end{pmatrix}.$$

As in previous simulations, the parameter ranges are chosen as $0 < \beta_i < 1$ and $0 < \alpha_i < 1\,000$, and a uniform distribution within the given range is used as the probabilistic model for the parameters. Capacities of the links are varied exponentially from 10^2 to 10^6. Results of the simulation are shown in Fig. 9. We observe that probability of stability increases with

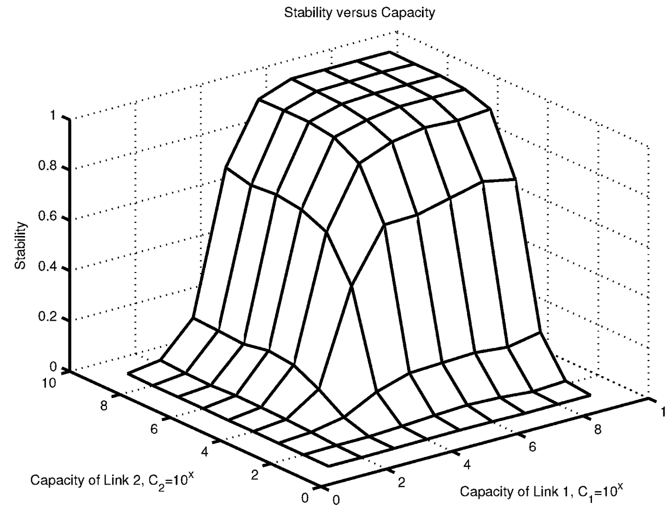

Fig. 9. Probability of stability for various capacities of links for the network in Example V.B.2.

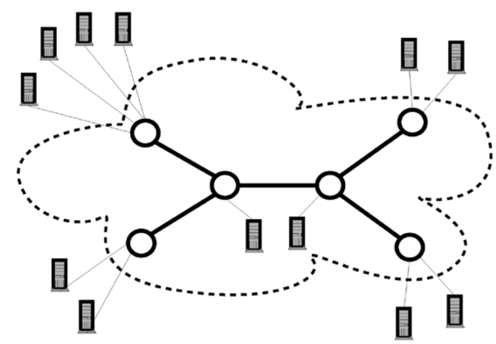

Fig. 10. Network topology with 12 users and five links with capacities $[35\ 50\ 30\ 15\ 20]10^3$.

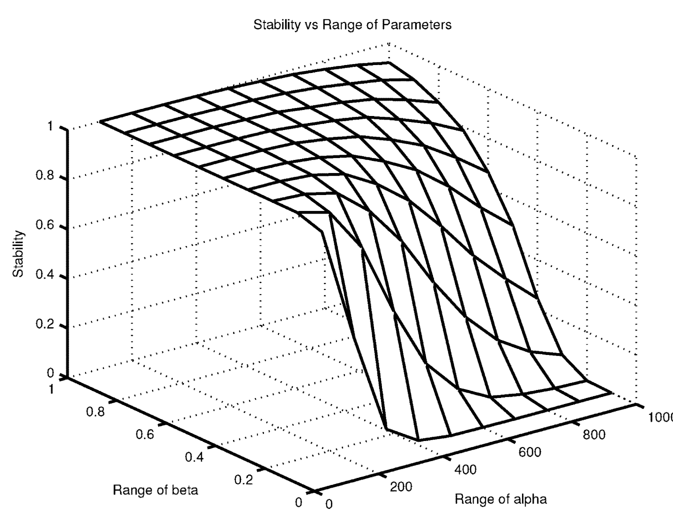

Fig. 11. Network stability for various ranges of parameters under the arbitrary network topology given by Fig. 10.

increasing capacity of the links, which is consistent with earlier results on the single bottleneck link case.

3) Simulations Under General Network Topologies: We next simulate the system under an arbitrary network topology

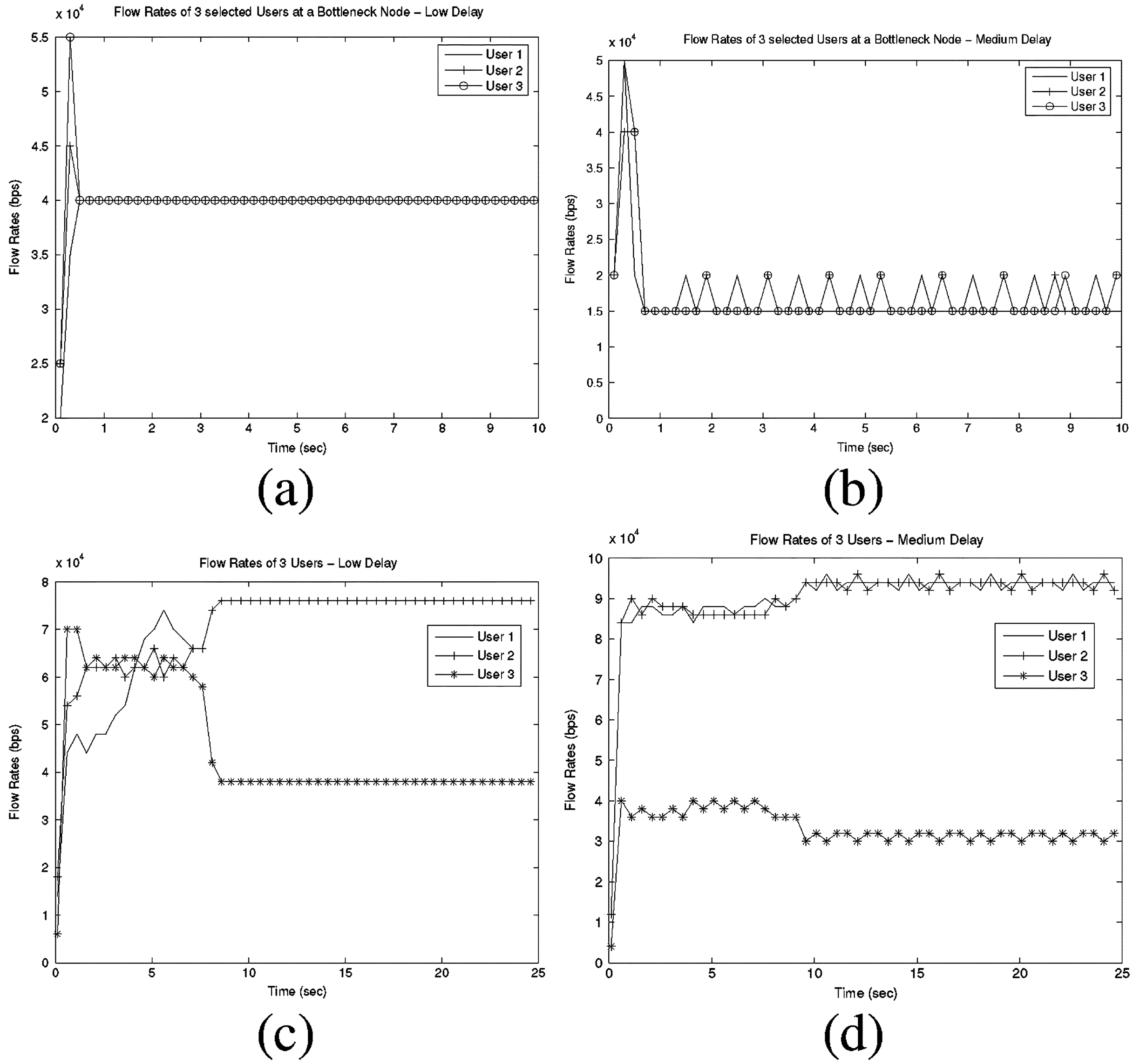

Fig. 12. *Ns-2* simulations depicting flow rates of selected users sharing a bottleneck link under (a) low and (b) medium delays. Flow rates of three users on the linear network of Fig. 8 under (c) low and (d) medium delays.

described by the routing matrix $\mathbf{A}$ given in the following with five links and 12 users:

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

It is in fact possible to repeat this simulation for arbitrarily large networks. The network structure adopted for this particular simulation is shown in Fig. 10. We investigate the local stability of the system for different parameter ranges varying from 0.1 to 1 for β and from 100 to 1 000 for α. A uniform distribution is used as probabilistic model within each given range of parameters. Capacities of the links $C_1, \dots, C_5$ are arbitrarily fixed to values $[35\ 50\ 30\ 15\ 20]10^3$. As a result of computational constraints, the number of samples is chosen to be $N = 10000$, which guarantees a level of confidence $\delta = 0.001$ and accuracy $\varepsilon = 0.02$. As we observe from Fig. 11, results are comparable to the ones obtained for the single bottleneck link case.

Finally, we investigate robustness of the linearized system to different routes. The number of users and links are the same as in the previous simulation. However, the routing matrix and, hence, the topology and routes in the network, are generated randomly in addition to the parameters which are generated using a uniform distribution within the ranges $0 < \beta_i < 1$ and $0 < \alpha_i < 1000, i = 1, \dots, 4$. The number of different topologies randomly generated is 50, and number of samples per topology is chosen as 5000 ensuring a level of confidence $\delta = 0.001$ and accuracy $\varepsilon = 0.03$. We observe that the system is stable with a probability of 0.70. If we increase the capacities of the links tenfold to $[35\ 50\ 30\ 15\ 20]10^4$, however, the probability of stability increases to 0.99. These observations are consistent with earlier simulation and analytical results obtained.

As noted earlier, any simulation in the $u - \alpha$ space is not feasible under general network topologies with multiple bottleneck links, as the explicit calculation of the unique equilibrium state requires the solution of a set of nonlinear equations.

C. Packet Level Simulations

We investigate and demonstrate the results observed in numerical simulations through realistic packet level simulations using the *ns-2* network simulator [34]. The model considered in this paper does not take feedback delays into account in order to maintain tractability of the already complicated model and be able to provide analytical results. On the other hand, we do realize that feedback delays play a nonnegligible role in network control problems [4], [15], [35]. Accordingly, we investigate the effects of feedback delays on stability by simulating both the low (1 ms) and medium (10 ms) delay scenarios on the *ns-2* simulator. We first simulate the case of a single bottleneck link of capacity $C = 10^6$ shared by 20 identical users with parameters $\alpha = 0.5$ and $u = 20000$, which are consistent with earlier parameter choices. We observe in Fig. 12(a) that the system is stable under 1-ms delay consistent with earlier simulation results. While the system is unstable under 10-ms delay as shown in Fig. 12(b) the variations in flow rates are small enough for practical purposes. Next, we simulate the linear network (Fig. 8) with three users analyzed in the illustrative Example V.B.2. The parameters are chosen as $\alpha = 500$ and $u = 10000$ similar to the ones in the example. Both of the link capacities are $C_1 = C_2 = 10^5$. The flow rates of users are depicted in Fig. 12(c) for low (1 ms) link delays and Fig. 12(d) for medium (10 ms) link delays, respectively. Again the results are consistent with the ones in the illustrative Example V.B.2 although some limited fluctuations are observed under medium delay. Thus, these packet level *ns-2* simulations demonstrate and are consistent with the previous analytical and numerical results based on fluid approximations.

VI. Conclusion

In this paper, we have investigated the local stability and robustness of a DT nonlinear congestion control algorithm, first at a single bottleneck link and then under general network topologies. For symmetric users at a single bottleneck link, we have obtained necessary and sufficient conditions for the local stability of the system. For more general scenarios, which include multiple bottleneck links and nonsymmetric users, analytic derivation is not possible and, hence, we have resorted to randomized algorithms and made use of both Monte Carlo and quasi-Monte Carlo methods. Specifically, we have used Halton, Niederreiter, and Sobol sequences as quasi-random sequences in addition to uniform random distributions created using standard pseudorandom number generators. As the quasi-random number generator implementations that we have obtained from various resources have dimension restrictions, we have used Monte Carlo methods for the analysis of systems with higher dimensions, corresponding to higher number of users.

This paper reveals that randomized algorithms provide extensive insight into local stability of congestion control algorithms which are inherently nonlinear. Furthermore, one can obtain accurate results on stability margins even with a small number of samples. One reason for this is that the linearized system has relatively simple stability boundaries in the parameter space as indicated by the analytical results we have obtained.

The robustness of the linearized system with respect to capacity at the bottleneck link and user parameters is investigated in two different parameter spaces, $\alpha - \beta$ and $u - \alpha$. The results are similar (though not identical), as to be expected due to the fact that each parameter space is a nonlinear transformation of the other one as given by the equations of the equilibrium state. As the actual probabilistic distribution of user parameters is unknown, both analyzes give important, independent insights into the robustness of the system. The simulation results for the multiple bottleneck link case are similar to the ones for the single bottleneck link case. In these simulations, we have investigated robustness of the system under various topologies and the interaction between system parameters and the routing matrix. We have also demonstrated the results obtained through packet level *ns-2* simulations and studied the effect of model assumptions to realistic implementations.

We have observed in Section V that under a given network topology, the nonlinear system is locally stable only for a specific range of values for the parameters, β, u, α, and C. The relationship between these parameters can also be interpreted from a network and congestion control perspective. We see that the relative magnitudes of utility parameter u and feedback (pricing) parameter α plays a significant role in determining system stability. For a given capacity C, if the parameter u is relatively high corresponding to aggressive user demand or pricing term α is relatively low then each case leads to instability due to ineffective feedback in the system. The variations in capacity C can be interpreted in a similar way as very low capacity magnifies the effect of u or user demand while very high capacity decreases the magnitude of the pricing term which is inversely proportional to capacity. On the other hand, the parameter β is proportional to u and inversely proportional to C^2 and, hence, more sensitive in capacity variations than changes in u.

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