Dynamic Pricing and Queue Stability in Wireless Random Access Games

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Abstract

We study the interaction among users of contention-based wireless networks, where the performance of the network is highly correlated with user transmission probabilities. Considering the underlying user incentives, we make use of the conceptual framework of noncooperative game theory to obtain a distributed control mechanism to limit the contention among wireless nodes by taking into account queue stability and injecting linear pricing to punish greedy behavior. We present a comprehensive analysis of the game including existence and uniqueness of Nash equilibrium point, convergence dynamics, and robustness properties.

Utilizing linear pricing enables us to move the equilibrium point of the game to a desirable region. We obtain conditions on linear prices necessary to achieve stability of user queues in the asymmetric and symmetric cases. In addition, we propose dynamic pricing algorithms, in which wireless users play the game without cooperation while the base station adjusts the linear price of each user. Under limited knowledge of game parameters, we present a dynamic equal pricing algorithm that moves the Nash equilibrium to the aggregate throughput maximizing solution. The theoretical results are verified, and the convergence and efficiency of the proposed game are illustrated via simulations.

Index Terms

Game theory, queue stability, adaptive pricing, Aloha, Nash equilibrium.

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I. Introduction

Contention-based medium access control plays a significant role in the successful deployment of modern wireless networks, where users are expected to manage their resources in a decentralized fashion. In such decentralized settings, selfishness of the users may lead to inappropriate utilization of resources and poor overall performance. When the transmission probabilities of the users are high, collisions dominate the system which in turn degrade the network throughput. In the opposite extreme, when the transmission probabilities are low, there are many idle slots reducing the utilization of the wireless channel. In this context, recent focus is to design optimization algorithms for contention based networks in the presence of selfish users with each user aiming to maximize its own performance [1], [2].

In the presence of selfish wireless nodes, game theory appears as a natural modeling framework, since it provides incentive compatibility to optimization, and networking problems. In this paper, we propose a noncooperative game that achieves a steady state with desirable features such as finite and stable user backlogs and/or maximum aggregate throughput. In this game, the strategy of a wireless user is the selection of its channel access probability, and the cost of this strategy is a function of its utility gain, channel access price, and queue size. The modeling of queue stability within such a game theoretical framework in contention based wireless networks constitutes one of the contributions of our work. We characterize the equilibrium point of the proposed game and investigate its existence and uniqueness properties. Furthermore, we study dynamical and distributed algorithms for computing and achieving the equilibrium solution. Specifically, we propose a gradient-based algorithm and show that it converges to the equilibrium point.

It may seem that selfish users applying their own strategies could lead to poor performance by constantly colliding in an attempt to maximize their individual throughput. However, the system performance depends on the cost associated with the users' transmissions [3]. Specifically, we show that the stability of user queues can be guaranteed by placing a bound on the linear channel access prices. In addition, we propose a dynamic equal pricing algorithm that does not require full knowledge and maximizes the aggregate throughput while providing fairness.

The contributions of this paper can be summarized as follows:

1) We design a noncooperative game in a slotted ALOHA setting, and analyze the existence and uniqueness of the Nash Equilibrium (NE) solution.
2) The convergence of the gradient based algorithm to this equilibrium is proven.
3) We obtain an implicit relationship between the individual channel access prices of the users that

needs to be satisfied for the stability of user queues.

4) Under a partial knowledge assumption and under the use of equal prices, we determine an adaptive pricing algorithm that shifts the NE point to the point where the maximum throughput is achieved.
5) We analyze the robustness of the game with respect to changes in prices and achievable channel rates.

The rest of the paper is organized as follows. Section II provides a brief summary of the earlier studies. Section III describes the system model and the random access game formulation. In Section IV, the existence of a unique Nash equilibrium solution and convergence properties of the game are established under some sufficient conditions. Then, a condition on the resource price is derived so that user queues are stable. In Section V, how the Nash equilibrium can be guided to a desired operating point is analyzed such as throughput maximal point. In Section VI, the robustness of the game with respect to linear pricing factor and channel capacity is studied. Lastly, Section VII contains the simulation results on the convergence of the game including adaptive pricing algorithm. Section VIII concludes this work by summarizing the contributions and discussing future directions.

II. Related Work

A plethora of work have emerged on the issue of optimizing the medium access control mechanism, especially for the slotted ALOHA systems. Here, we restrict ourselves to cite a few that are most closely related to our work, i.e., we focus on optimization of medium access mechanisms using game theory. In [1], a stability region has been obtained for a slotted multi-packet Aloha system with selfish users, perfect information, and under the assumption of some well-known channel models. [2] has considered both the cooperative team problem as well as the noncooperative game problem to minimize the delay in slotted ALOHA. Unlike these works, [4] and [5] have studied distributed choices of transmission probabilities in the slotted Aloha with partial information with imposing priorities and random power. [6] has studied noncooperative equilibria of Aloha networks and their local convergence.

In addition to aforementioned works that focused on the slotted ALOHA medium access mechanism, other MAC mechanisms have also been investigated. [7] has discussed selfish behavior in CSMA/CA networks using game theoretical approach and proposed a distributed protocol to guide the selfish nodes to a Pareto-optimal Nash equilibrium. [8] has investigated the interaction among wireless nodes in a game theoretical framework and designed medium access methods that can stabilize the network around a steady state with a target fairness and high efficiency. [9] studies cooperation in multiple-access networks via coalitional game theory. They consider two strategies by which nodes cooperate to avoid collision or idle times within a coalition. Consequently, a grand coalition is formed to optimize the sum rate.

Our design and analysis differ in terms of both considering the stability of user queues and implementing a dynamic pricing algorithm to achieve some specified goals such as maximizing throughput and stabilizing user queues.

III. SYSTEM MODEL

We consider an uplink wireless network where N users are competing to gain access to a single base station (BS). The time is slotted, and the length of the time slot is equal to the channel coherence interval. The achievable transmission rate of user i at time slot t is $C_i(t)$. The medium access mechanism is basic slotted ALOHA [10], where each user i attempts to transmit with some probability $q_i(t)$. ALOHA is a medium access mechanism which is sufficiently simple to analyze but at the same time that is sufficiently generic to draw meaningful conclusions. Also, some form of ALOHA protocol is used in many different advanced wireless access protocols, e.g., cellular networks.

In this work, we assume that the network consists of selfish users where each user aims to maximize its net benefit while keeping its queue stable at the same time. The net benefit of a user is defined as the difference between the utility obtained from the network access and the cost of this access. The utility function is taken to be a non-decreasing concave function of the throughput. This choice is of practical interest, since a small increase in the rate in the low rate regime is generally more appreciated than a small increase in the high rate regime. In accordance with most prior works, from now on, we assume a logarithmic utility function. The rate of change in logarithmic utility function of a user solely depends on the strategy choice of that user, not all the strategy space. This property allows us to obtain conditions on how the prices should be selected or how many users should be in the system.

When interactions between the users are taken into account, game theory emerges as a natural modeling framework. In this paper, we design a noncooperative game model, in which users not only aim to maximize their net benefits but also minimize their queue backlogs. The base station participates in the game by controlling the price of wireless medium access attempts. The players of the game are the users of the network, and their actions are defined by their transmission probabilities, q_i. We assume that each participating user is active, i.e. transmits at least with a very small but nonzero probability ϵ. The base station controls the game by varying the transmission prices, k_i for each user i. At each time slot t, the following game is played.

Definition 1. *The stabilizing random access game in slotted ALOHA system is defined as*

$$\Gamma \triangleq \{\mathcal{N}, (s_i)_{i\in\mathcal{N}}, (J_i)_{i\in\mathcal{N}}\}, \tag{1}$$

where $\mathcal{N}$ is the set of N wireless users in the network; $s_i \triangleq \{q_i : q_i \in [\epsilon, 1]\}$ is the strategy space corresponding to transmission probabilities of users; and J_i is the cost of strategy of user i.

Let us define the user cost function as the sum of three terms, where each term identifies an important aspect of the system model. The first term is the utility achieved by the user and is assumed to be related to the log throughput, i.e.,

$$J_i^1(t) = -\eta_i \log\left[C_i(t)q_i(t)\prod_{j\neq i}(1-q_j(t))\right]. \tag{2}$$

In (2) $\eta_i > 0$ represents the preference of the different types of users for network throughput[1].

The second term penalizes the positive drift in the queue size, and it is utilized to achieve the stability of user queues.

$$J_i^2(t) = \Delta B_i(t). \tag{3}$$

In (3), $\Delta B_i(t)$ is defined as the drift in the size of the queue of user i which is obtained by subtracting the service rate from the arrival rate:

$$\Delta B_i(t) = A_i(t) - C_i(t)q_i(t)\prod_{j\neq i}(1-q_j(t)),$$

where $A_i(t)$ is the arrival rate of packets for user i at time t.

The final term represents the cost of access to the channel, and it can be interpreted as the punishment of the greedy behavior:

$$J_i^3(t) = k_i q_i(t). \tag{4}$$

In (4), $k_i > 0$ is the linear price of channel access attempt.

Overall, we have the following aggregate cost function:

$$J_i(q_i, \mathbf{q}_{-i}) = k_i q_i(t) + \Delta B_i(t) - \eta_i \log\left[C_i(t)q_i(t)\prod_{j\neq i}(1-q_j(t))\right]. \tag{5}$$

[1]The negative sign here is due to the cost minimization convention adopted in this paper as opposed to utility maximization.

Given the strategy vector, $\mathbf{q}_{-i}$ of all other users[2], i.e., $\mathbf{q}_{-i} \triangleq (q_1, ..., q_{i-1}, q_{i+1}, ..., q_N)$, each user i aims to solve the following optimization problem:

$$\min_{q_i} J_i(q_i, \mathbf{q}_{-i}) \tag{6}$$

We note that the players (users) are tightly coupled with each other in the sense that the actions (transmission probabilities) of each player affect the performance of others significantly. In fact, a single player can even block access to whole channel single-handedly if $q = 1$. Unsurprisingly, overall throughput of users can only be achieved if the transmission probabilities decrease proportional to the number of users sharing the channel. The selfish nature of the players may prevent obtaining (such) a mutually beneficial solution by themselves as a result of a phenomenon well-known as "tragedy of commons", i.e., the (Nash) equilibrium outcome of the game being very undesirable. Pricing schemes are utilized in such cases to improve the outcome of the game. In the next section, the (Nash equilibrium) solution of the game and its properties are analyzed.

IV. Equilibrium and Stability Analysis

A. Existence and Uniqueness of Nash Equilibrium

One of the fundamental issues in the random access game is the analysis of equilibrium solutions, especially existence and uniqueness of a Nash equilibrium (NE) solution. The definition of the NE is provided below for completeness.

Definition 2. *A strategy vector, i.e. transmission probabilities of wireless users, $\mathbf{q}^*$, of the random access game Γ defined in (1) is said to be in Nash Equilibrium if no user can improve its cost function by deviating from Nash Equilibrium point [11].*

$$J_i(q_i^*, \mathbf{q}_{-i}^*) \leq J_i(q_i, \mathbf{q}_{-i}^*), \ \forall q_i \in s_i. \tag{7}$$

In game Γ, since the user cost function is convex, the NE solution may exist at the intersection of the player best-responses [11], which follow from the first-order optimality condition:

$$\begin{aligned} &\frac{\partial}{\partial q_i} J_i(q_i, \mathbf{q}_{-i}^*)|_{q_i = q_i^*} = 0, \ \forall i \\ &\Rightarrow k_i - C_i \prod_{j \neq i} (1 - q_j^*) - \frac{\eta_i}{q_i^*} = 0, \ \forall i. \end{aligned} \tag{8}$$

[2]We are going to first analyze the game in every slot, and hence we omit the time parameter t for brevity.

Hence, the NE transmission probability, q_i^*, is

$$q_i^* = \left[\frac{\eta_i}{k_i - C_i \prod_{j \neq i}(1 - q_j^*)} \right]_\epsilon^1, \tag{9}$$

where $[\cdot]_\epsilon^1 = \min(\max(\epsilon, \cdot), 1)$, for some $\epsilon > 0$, i.e., the value is bounded above and below, respectively.

As described in (9), the transmission probability at the equilibrium of user i decreases in the linear price, k_i, and the transmission probability of other users, $q_j^*, \ \forall j \neq i$. On the other hand, an increase in the achievable transmission rate, C_i, results in an increase in q_i^*.

Next, we analyze the existence and uniqueness of the Nash Equilibrium. Our analysis applies the results given in [12] and [13] for our particular game.

Lemma 1. *The strategy space of the game Γ, defined in (1), $\mathcal{Q} = [\epsilon, 1]^N \subset \mathbb{R}^N$, is convex, compact, and has a nonempty interior, provided that $\epsilon < 1$.*

Lemma 2. *The cost function of the i^{th} player, J_i in (5), is twice continuously differentiable and strictly convex in q_i, i.e., $\partial^2 J_i / \partial q_i^2 > 0$, and $\partial^2 J_i / \partial q_i \partial q_j > 0$ on $\mathcal{Q}$.*

Proof Note that $\eta_i > 0$ and $k_i > 0$. The second partial derivatives of cost function, J_i, are $\frac{\partial^2 J_i}{\partial q_i^2} = \frac{\eta_i}{q_i^2}$, and $\frac{\partial^2 J_i}{\partial q_i \partial q_j} = C_i \prod_{n \neq i,j}(1 - q_n)$. Thus, $\frac{\partial^2 J_i}{\partial q_i^2}$ and $\frac{\partial^2 J_i}{\partial q_i \partial q_j}$ are always greater than zero on $\mathcal{Q}$. ■

Proposition 1. *The random access game, Γ defined in (1), admits at least one NE solution.*

Proof According to Lemma 1 and 2, J_i is a differentiable convex function. Hence, $\frac{\partial J_i}{\partial q_i}$ is a continuous and increasing function, and so its inverse $\left(\frac{\partial J_i}{\partial q_i}\right)^{-1}$ is also continuous. Note that the range of transmission probabilities, $[\epsilon, 1]$, is a connected and compact set. Therefore, the range of $\left(\frac{\partial J_i}{\partial q_i}\right)^{-1}$ is also a connected and compact set. Based on Theorem 4.4 in [12] and Theorem 1 in [13], we can conclude that there exists at least one NE solution. ■

Next, we investigate the conditions under which the game admits a unique NE solution. Let $g(\mathbf{q}) = [g_1(\mathbf{q}), ..., g_N(\mathbf{q})]$ where $g_k(\mathbf{q}) = \frac{\partial J_k}{\partial q_k}, \ k = 1, ..., N$ and $G(\mathbf{q})$ be the Jacobian of $g(\mathbf{q})$ which is defined as:

$$G(\mathbf{q}) = \begin{bmatrix} b_1 & a_{12} & \dots & a_{1N} \\ a_{21} & b_2 & \dots & a_{2N} \\ \vdots & & \ddots & \vdots \\ a_{N1} & a_{N2} & \dots & b_N \end{bmatrix}, \tag{10}$$

where $b_i = \partial^2 J_i/\partial q_i^2$ and $a_{ij} = \partial^2 J_i/\partial q_i \partial q_j$.

Definition 3. *A matrix is said to be strictly diagonally dominant if in every row of the matrix, the magnitude of the diagonal entry in that row is larger than the sum of the magnitudes of all the other (non-diagonal) entries in that row.*

In our case, the matrix $G(q)$ is strictly diagonally dominant if $|b_i| > \sum_{j \neq i} |a_{ij}|, \ \forall i$.

Lemma 3. *If* $q_i < \sqrt{\frac{\eta_i}{C_i(N-1)}}$, $\forall i$, *then* $G(q)$ *is strictly diagonally dominant.*

Proof

$$q_i < \sqrt{\frac{\eta_i}{C_i(N-1)}} \tag{11}$$

$$\frac{\eta_i}{q_i^2} > C_i(N-1) \tag{12}$$

$$\frac{\eta_i}{q_i^2} > \sum_{j \neq i} \left(C_i \prod_{n \neq i,j} (1 - q_n) \right), \tag{13}$$

$$b_i > \sum_{j \neq i} a_{ij}, \tag{14}$$

where (13) follows from $(1 - q_n) \leq 1$, and (14) is from the proof of Lemma 2. ■

The following result, which is a variation of Theorem 2.1 in [14], is also needed to further the analysis.

Lemma 4. *[15] A strictly diagonally dominant matrix is non-singular.*

We next establish the uniqueness of an **inner** NE solution on a subset of the strategy space $\mathcal{Q}$:

Theorem 1. *The random access game* Γ *of Definition 1 admits an inner Nash equilibrium solution that is unique on the strategy space*

$$\bar{\mathcal{Q}} := \bigotimes_{i=1}^{N} \left(\epsilon, \sqrt{\frac{\eta_i}{C_i(N-1)}} \right), \tag{15}$$

where $\bigotimes$ *denotes cross-product of the interval sets.*

Proof Suppose that there are two inner equilibrium points, represented by $\mathbf{q}^1$ and $\mathbf{q}^0$. Define the strategy vector $\mathbf{q}(\theta)$ as a convex combination of the two equilibrium points $\mathbf{q}^1$, $\mathbf{q}^0$:

$$\mathbf{q}(\theta) = \theta \mathbf{q}^1 + (1 - \theta)\mathbf{q}^0, \tag{16}$$

where $0 < \theta < 1$. Note that $\mathbf{q}(\theta)$ still satisfies the condition in the theorem. When we take the derivative of $g(\mathbf{q}(\theta))$ with respect to θ, we obtain

$$\frac{dg(\mathbf{q}(\theta))}{d\theta} = G(\mathbf{q}(\theta))\frac{d\mathbf{q}(\theta)}{d\theta} = G(\mathbf{q}(\theta))(\mathbf{q}^1 - \mathbf{q}^0). \tag{17}$$

By integrating (17) over θ yields,

$$g(\mathbf{q}^1) - g(\mathbf{q}^0) = \left[\int_0^1 G(\mathbf{q}(\theta))d\theta\right](\mathbf{q}^1 - \mathbf{q}^0). \tag{18}$$

Recall that $\mathbf{q}^1$ and $\mathbf{q}^0$ are equilibrium points, so $g(\mathbf{q}^1) = 0$ and $g(\mathbf{q}^0) = 0$. From Lemma 3, we know that $G(\mathbf{q}(\theta))$ is strictly diagonally dominant, and hence, $\int_0^1 G(\mathbf{q}(\theta))d\theta$ is strictly diagonally dominant and it is also non-singular based on Lemma 4. Thus, the matrix, $\int_0^1 G(\mathbf{q}(\theta))d\theta$, is also non-singular. Then, it is clear that (18) is equal to zero, only when $\mathbf{q}^1 - \mathbf{q}^0 = 0$. Therefore, there cannot be more than one equilibrium point. ■

Next, we include the **boundary solutions** of the game in our investigation:

Theorem 2. *The random access game* Γ *of Definition 1 admits a unique inner Nash equilibrium solution on the strategy space*

$$\tilde{\mathcal{Q}} := \bigotimes_{i=1}^{N}\left[\epsilon, \sqrt{\frac{\eta_i}{C_i(N-1)}}\right], \tag{19}$$

if ϵ *is chosen sufficiently small and*

$$k_i > C_i + \sqrt{C_i\eta_i(N-1)} \ \ \forall i.$$

Proof Notice that for a sufficiently small ϵ the players can always improve their performance by increasing their transmission rate when they transmit at the lower boundary, which follows directly from (8). Likewise, the sufficient condition on individual prices k_i ensures that $\partial J_i/\partial q_i > 0$ at the upper boundary points. Thus, any boundary solution on $\bar{\mathcal{Q}}$ cannot constitute a Nash equilibrium solution (by definition of NE). The rest of the proof follows directly from the one of Theorem 1. ■

B. Convergence to Nash Equilibrium under Gradient Algorithm

Once we establish that there is a unique Nash equilibrium solution under certain conditions, the next question is to determine a distributed algorithm achieving this solution, since the unique Nash equilibrium solution given in (9) cannot be explicitly found.

We consider a dynamic transmission probability update mechanism, where each user utilizes a gradient algorithm to solve its own optimization problem within a given time slot. Let us assume that user i updates its transmission probability according to the following dynamic equation:

$$\frac{\mathrm{d}\, q_i}{\mathrm{d}\, t} = -\frac{\partial J_i}{\partial q_i} =: \phi_i, \tag{20}$$

where

$$\phi_i = \left(-k_i + C_i \prod_{j \neq i} (1 - q_j) + \frac{\eta_i}{q_i} \right). \tag{21}$$

Algorithm 1 summarizes the algorithmic implementation of the discrete-time counterpart of the dynamic system defined by the differential equation (20).

Algorithm 1 Gradient Update Algorithm

1) Initialize $q_i^{(0)}$, $\forall i$ and set $l = 0$.
2) Iteratively update $q_i^{(l+1)}$ as follows:

$$q_i^{(l+1)} = \left[q_i^{(l)} + \lambda \left(-k_i + C_i \prod_{j \neq i} (1 - q_j^{(l)}) + \frac{\eta_i}{q_i^{(l)}} \right) \right]_{\epsilon}^{1},$$

where $\lambda > 0$ is a sufficiently small step size.
3) Repeat (2) until convergence is realized, i.e, $|q_i^{(l+1)} - q_i^{(l)}| < \varsigma \; \forall i$ where ς is a small positive number.

Both the dynamic system in (20) and Algorithm 1 defines a method for user i to update its transmission probability based on the channel capacities and the transmission probabilities of the other users in the system. This update is in the opposite direction of the gradient of the total cost.

Note that both the users and the base station has sufficient information to implement the algorithm. The users need to measure their own throughput in order to update their transmission probabilities. Likewise, by measuring the throughput of individual users, the base station can compute their transmission probabilities without asking them explicitly, and hence impose prices. The only information exchange, other than measurements, in the system is due to base station telling users their individual prices k_i.

We next show that the dynamic update mechanism described by (20) is asymptotically stable, and hence converges to the unique inner NE of the game.

Theorem 3. *Let* $q^{NE} \triangleq [q_1^*, q_2^*, \ldots, q_N^*]$ *be the unique inner NE of the game* Γ*, defined in (1), with the strategy space* $\tilde{\mathcal{Q}}$ *in* (19)*. The system dynamics stated in* (20) *are asymptotically stable, and converge to* *the unique NE, under the sufficient conditions of Theorem 2, i.e. if ϵ is chosen sufficiently small and*

$$k_i > C_i + \sqrt{C_i \eta_i (N-1)} \;\; \forall i.$$

Proof

The proof can be found in the Appendix. ■

Until now, we have analyzed the properties of the Nash equilibrium solution and shown its uniqueness and distributed computation by the users. In the following section, we aim to answer the following question: "How should the prices, k_i, be selected so that some global objectives such as queue stability and utility maximization are satisfied?"

C. Stability of User Queues

In this section, we investigate how the pricing parameter, k_i should be chosen so that the user queues are stabilized. We obtain an implicit relationship between k_i's that needs to be satisfied to ensure that queue drift in each queue is zero, i.e., $\Delta B_i^* = 0$, when each user has potentially different set of system parameters and costs. Secondly, we obtain closed from solution of k for symmetric two user case, where the users are indistinguishable in terms of their cost and system parameters.

In order for a queue to be stable at the equilibrium, the drift of the queue should be negative or zero, i.e.,

$$\Delta B_i^* = A_i - C_i q_i^* \prod_{j \neq i} (1 - q_j^*) \leq 0. \tag{22}$$

If the above is satisfied with equality, the queue size neither decreases nor increases. In this case, one may assume that there is always a packet in the queue. Here, we give analysis when $\Delta B_i^* = 0$.

Inserting equilibrium solution, q_i^*, in (9) into the drift equation in (22), we obtain

$$\prod_{j \neq i} (1 - q_j^*) = \frac{A_i k_i}{(A_i + \eta_i) C_i}. \tag{23}$$

Let $\alpha_i = \log(1 - q_i^*)$ and $\beta_i = \log\left(\frac{A_i k_i}{(A_i + \eta_i) C_i}\right)$. Then, from (23), we obtain $\sum_{j \neq i} \alpha_j = \beta_i$. Note that the transmission probabilities vary with respect to k, which is the variable used to punish the greedy behavior. Also as shown in (23), the value of k affects the stability of the queues.

Let us define matrix E as a matrix with all entries except diagonal ones are equal to 1, i.e.,

$$E = \begin{bmatrix} 0 & 1 & \cdots & 1 & 1 \\ 1 & 0 & \cdots & 1 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & \cdots & 0 & 1 \\ 1 & 1 & \dots & 1 & 0 \end{bmatrix}.$$

Also, define $d = [\alpha_1, \alpha_2, \dots, \alpha_N]$ and $b = [\beta_1, \beta_2, \dots, \beta_N]$. Then, (23) can be re-written in matrix form as:

$$\begin{aligned} E \cdot d^T &= b^T \\ d^T &= E^{-1} \cdot b^T. \end{aligned} \tag{24}$$

Note that E^{-1} is defined as:

$$E^{-1} = \begin{bmatrix} \frac{2-N}{N-1} & \frac{1}{N-1} & \cdots & \frac{1}{N-1} & \frac{1}{N-1} \\ \frac{1}{N-1} & \frac{2-N}{N-1} & \cdots & \frac{1}{N-1} & \frac{1}{N-1} \\ \vdots & & \ddots & & \vdots \\ \frac{1}{N-1} & \cdots & \frac{1}{N-1} & \frac{2-N}{N-1} & \frac{1}{N-1} \\ \frac{1}{N-1} & & \cdots & \frac{1}{N-1} & \frac{2-N}{N-1} \end{bmatrix}. \tag{25}$$

Since (23) should be satisfied so that the user queues are stable, the i^{th} row of $E^{-1} \cdot b^T$ should be equal to the i^{th} row of d^T:

$$\alpha_i = \frac{1}{N-1} \left(\beta_i (2 - N) + \sum_{j \neq i} \beta_j \right). \tag{26}$$

After some mathematical manipulations, we obtain the following equation of the equilibrium transmission probability that ensures the stability of the queues as:

$$q_i^* = 1 - \left(\frac{A_i k_i}{(A_i + \eta_i) C_i} \right)^{\frac{2-N}{N-1}} \prod_{j \neq i} \left(\frac{A_j k_j}{(A_j + \eta_j) C_j} \right)^{\frac{1}{N-1}} \tag{27}$$

Unfortunately, the closed form solution of k_i cannot be obtained from (27) due to the nonlinear structure of the inequality.

Aforementioned analysis can be extended to the case when the queue drifts are negative. When the queue drifts are negative, the user queues tend to get empty. Hence, some of the users do not have sufficient number of packets to transmit and it is not possible to guarantee an inner point Nash equilibrium solution. In fact, those users with empty queues are no longer part of the game, since they cannot transmit with positive probability. As users with empty queues are out of the game, the game is played only among those players with non-empty queues. This new game can be analyzed in exactly the same way as discussed before, but obviously the game has fewer number of players. In this case, both the Nash equilibrium transmission probabilities and the access prices for ensuring negative queue drift need to be re-calculated according to our results given in earlier sections. As the queues get empty and get filled up again, the whole analysis have to be repeated.

Note that, the queues in the system cannot be stabilized for all arrival rates. Thus, a closed form solution can give us intuition through which arrival rates, A, and prices, k, the stability of the queues can be realized. Thus, we now consider two user symmetric case, i.e., $A_1 = A_2 = A$, $C_1 = C_2 = C$ and $\eta_1 = \eta_2 = \eta$.

By solving (27) and equilibrium solution in (9) simultaneously, we obtain the following prices, k, which drives queue drift to zero:

$$k = \frac{0.5(C \mp \sqrt{C^2 - 4AC})(A + \eta)}{A}. \tag{28}$$

It is easy to observe that the user queues are stable only for $A < \frac{C}{4}$. For symmetric case, the maximum rate is achieved when $q^* = 1/2$, which corresponds to a maximum achievable rate of $C/4$. Thus, the condition $A < \frac{C}{4}$ suggests that unless the arrival rate, A, is smaller than the maximum achievable rate, the queues cannot be stabilized.

V. Optimization Through Pricing

Arbitrary choice of game parameters can lead to a loss of efficiency in the game discussed. Thus, selecting the prices carefully can significantly improve the efficiency of Nash Equilibrium with respect to a chosen criterion in specific settings, e.g., [16], [17]. In [18], conditions for pricing functions which allow locating the NE solutions to any desired point, are derived. In a similar spirit, we investigate how to move the equilibrium to maximum throughput point with partial network information. The general structure of the algorithms is presented in Figure 1.

A. Pricing for Global Optimization of User Utilities

In some practical systems, the transmission rates of the users, i.e., C_i, are known by the BS. In addition, users can be classified under certain classes, i.e., η_i may also be known. In such cases, the

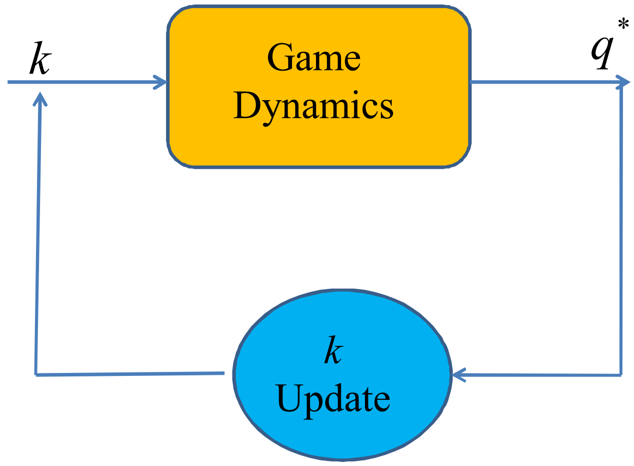

Fig. 1. Feedback control of the game using pricing parameter, k, as the control parameter.

optimal transmission probabilities can be obtained by the base station.

When our objective is to maximize the social welfare, i.e., the sum of user utilities, then the problem can be shown to be a convex optimization problem. Thus, one can numerically determine the optimal transmission probabilities, q_i^*, and hence, given the user utility functions, the BS can easily calculate the optimal prices k_i^*. The utility of user i is defined as the sum of J_i^1 in (2) and J_i^2 in (3):

$$U_i(\mathbf{q}) \triangleq \eta_i \log(C_i q_i \prod_{j \neq i}(1-q_j)) - \Delta B_i \tag{29}$$

Assume a social welfare function, $U(\mathbf{q})$, is defined as the sum of these concave utility functions, $U(\mathbf{q}) := \sum_i U_i(\mathbf{q})$.

Theorem 4. *If $\eta_i > C_i(N-1)$, $\forall i$, then the social welfare function, $U(\mathbf{q})$, is concave with respect to all q_i's, and the optimal transmission probability of user i, q_i^* is:*

$$q_i^* = \frac{x + \sum \eta_k \mp \sqrt{(x+\sum \eta_k)^2 - 4\eta_i x}}{2x} \tag{30}$$

where $x = \sum_{j \neq i} C_j \prod_{n \neq i,j}(1-q_n^*) - C_i \prod_{j \neq i}(1-q_j^*)$.

Proof Note that the utility function in (29) coincides with the cost function in (5) except the linear term, $k_i q_i$. Since linear terms do not have any effect on convexity or concavity, we conclude that the utility of user i is a strictly concave function under the same condition for convexity of the cost function stated in

Lemma 3 as $q_i < \sqrt{\frac{\eta_i}{C_i(N-1)}}$. However, since the optimization is performed with respect to transmission probabilities, the obtained condition should hold for all possible transmission probabilities. Thus, we modify the condition as $\eta_i > C_i(N-1)$ by inserting the upper limit of q_i, which is 1, instead of q_i. Lastly, the sum of these concave utility functions is again strictly concave.

Next, we obtain the optimum q_i^*. The first derivative of the social welfare function with respect q_i is as follows:

$$\frac{\partial U(\mathbf{q})}{\partial q_i} = \frac{\eta_i}{q_i} + C_i \prod_{j \neq i}(1-q_j) - \sum_{j \neq i}\left[\frac{\eta_j}{1-q_i} + C_j \prod_{n \neq i,j}(1-q_n)\right] \tag{31}$$

If we equate (31) to zero, and solve for q_i, we obtain q_i^* as:

$$q_i^* = \frac{x + \sum \eta_k \mp \sqrt{(x + \sum \eta_k)^2 - 4\eta_i x}}{2x} \tag{32}$$

where $x = \sum_{j \neq i} C_j \prod_{n \neq i,j}(1-q_n) - C_i \prod_{j \neq i}(1-q_j)$. ■

Thus, we derive the optimal pricing parameter as $k_i^* = C_i \prod_{j \neq i}(1-q_j^*) + \frac{\eta_i}{q_i^*}$.

B. Dynamic Equal Pricing for Throughput Maximization

In Section V-A, we have shown that the social welfare point can theoretically be obtained. However, in practice, implementing an update algorithm that reaches the social welfare point is not trivial due to lack of information about the users' preferences.

In practice, user preferences may be unknown, or they cannot be easily determined. Also, charging different users different access prices may not be operationally feasible or preferable. Thus, we next propose an adaptive pricing algorithm that maximizes the aggregate throughput with equal pricing, i.e., that each user is charged with the same price, k. The question is to find a method to select k in a dynamic network environment. If k is chosen too small, then q_i^* will be too high, and collisions will dominate the system. On the contrary, if k is chosen to be high, then q_i^* will be too small and channel will be under-utilized with many idle slots.

In this specific setting, we can determine whether k is too small or too high by simply observing the aggregate throughput curve such as the one illustrated in Fig. 2. The aggregate throughput is observed in simulations to be a unimodal function, and thus, we can determine the optimal point with different kinds of search algorithms, e.g., newton search algorithm, binary search or golden section search algorithms.

In any search algorithm, the idea is to iteratively select an approximate search interval which closes in onto the optimal value at each passing step. The search algorithm is implemented in our proposed

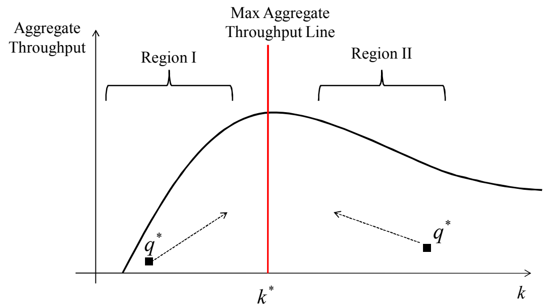

Fig. 2. Aggregate Throughput versus the pricing parameter, k

random access game in the following way. The base station selects an initial price which is announced to all users. Based on this selected price value, users start playing the game. After a number of slots, game reaches an equilibrium. The base station observes the channel for a certain duration and obtains the value of the aggregate throughput. Base station updates the price according the search algorithm implemented and the cycle continues until the length of the search interval is smaller than some small constant.

Even though search algorithms have high convergence rates, they may lack robustness. As network parameters vary, the optimal price also changes, and the search algorithm should be run again from scratch. Thus, if the network parameters vary often in short time scales, then search algorithms may perform poorly. Next, we investigate a robust dynamic pricing algorithm.

Robust Dynamic Pricing Algorithm: We propose the following robust dynamic pricing algorithm as an alternative to search schemes:

$$\dot{k} = \begin{cases} \gamma, & \text{if } k \in \text{ Region I} \\ -\mu, & \text{if } k \in \text{ Region II} \end{cases} \tag{33}$$

where μ and γ are update parameters. As seen in Fig. 2, in Region I, there is a sharper decrease in aggregate throughput. Thus, we want to emanate from this region as soon as possible, so we select $\gamma > \mu$.

Note that the base station can easily determine whether the users are in the Region I or Region II. In Region I, as the price, k is increased, the aggregate throughput should increase as well. On the other hand, In Region II, an increase in k leads to a drop in aggregate throughput. Once the base station determine the region, it announces the users whether it will increase or decrease k.

The aforementioned dynamic algorithm pushes the system to a region around the optimal price, k^*. However, the system does not converge to a single point as before but oscillates within a region whose size depends on the values of μ and γ. Utilizing larger values of μ and γ results in high convergence, but at the same time to a larger convergence region and more deviation from the optimal point, k^*. Adaptive algorithms may be considered to address this issue.

VI. Robustness Analysis

In this section, the effects of estimation errors on k_i and C_i are investigated, and it is shown that the distance to the optimal transmission probability, q_i^*, is bounded by a constant factor of the estimation error. Next theorem analyzes how an error in k affects the transmission probabilities, q_i^*.

Theorem 5. *Let the pricing parameter, k, be perturbed by a "small" scalar, ε. Then, the change in transmission probabilities, $\mathbf{q}^*$, is quantified by*

$$F \leq Z^{-1} \cdot Y \cdot |\varepsilon|,$$

where $f_i(\varepsilon) = q_i^*(k+\varepsilon) - q_i^*(k)\ \forall i$,

$$Z = \begin{bmatrix} 1 & -\frac{C_1}{\zeta_1} & \cdots & -\frac{C_1}{\zeta_1} \\ -\frac{C_2}{\zeta_2} & 1 & \cdots & -\frac{C_2}{\zeta_2} \\ \vdots & \vdots & \ddots & \vdots \\ -\frac{C_N}{\zeta_N} & -\frac{C_N}{\zeta_N} & \cdots & 1 \end{bmatrix},$$

$F = [|f_1(\varepsilon)|, |f_2(\varepsilon)|, \ldots, |f_N(\varepsilon)|]^T$,
$Y = \left[\frac{1}{\zeta_1}, \frac{1}{\zeta_2}, \ldots, \frac{1}{\zeta_N}\right]^T$, *and* $\zeta_i = k - C_i \prod_{j \neq i}(1 - q_j^*(k))$.

Proof The change in transmission probabilities with respect to price, k, is in the same direction for all users. Specifically, $f_i(\varepsilon) < 0$ if $\varepsilon > 0, \forall i$ and $f_i(\varepsilon) > 0$ if $\varepsilon < 0, \forall i$. One can show that the following inequality holds:

$$
\begin{aligned}
&\left|\prod_{j\neq i}(1-q_j^*(k+\varepsilon))-\prod_{j\neq i}(1-q_j^*(k))\right| \\
&=\left|-\sum_{j\neq i}(q_j^*(k+\varepsilon)-q_j^*(k))\prod_{m<j,m\neq i}(1-q_m^*(k))\prod_{n>j,n\neq i}(1-q_n^*(k+\varepsilon))\right|, \\
&=\left|-\sum_{j\neq i}f_j(\varepsilon)\prod_{m<j,m\neq i}(1-q_m^*(k))\prod_{n>j,n\neq i}(1-q_n^*(k)-f_n(\varepsilon))\right|, \qquad (34)\\
&\leq \sum_{j\neq i}|f_j(\varepsilon)|, \qquad (35)
\end{aligned}
$$

where (35) follows from the fact that the product of terms in (34) is always less than one.

At equilibrium, the value of $f_i(\varepsilon)$ can also be given as:

$$
\begin{aligned}
f_i(\varepsilon) =&\frac{\eta_i}{k+\varepsilon-C_i\prod_{j\neq i}(1-q_j^*(k+\varepsilon))}-\frac{\eta_i}{k-C_i\prod_{j\neq i}(1-q_j^*(k))} \\
=&\eta_i\frac{C_i\left(\prod_{j\neq i}(1-q_j^*(k+\varepsilon))-\prod_{j\neq i}(1-q_j^*(k))\right)-\varepsilon}{\zeta_i(\varepsilon)\zeta_i} \qquad (36)
\end{aligned}
$$

where $\zeta_i(\varepsilon)=k+\varepsilon-C_i\prod_{j\neq i}(1-q_j^*(k+\varepsilon))$. Combining (36) and (35), we obtain

$$
\begin{aligned}
|f_i(\varepsilon)| &\leq \eta_i\frac{C_i}{\zeta_i(\varepsilon)\zeta_i}\sum_{j\neq i}|f_j(\varepsilon)|+\eta_i\frac{|\varepsilon|}{\zeta_i(\varepsilon)\zeta_i} \qquad (37)\\
&\leq \frac{C_i}{\zeta_i}\sum_{j\neq i}|f_j(\varepsilon)|+\frac{|\varepsilon|}{\zeta_i} \qquad (38)
\end{aligned}
$$

The last inequality follows from the fact that $\eta_i/\zeta_i(\varepsilon)$ is equal to the equilibrium transmission probability given in (9), and thus, it is less than 1. Re-writing (38) in matrix form yields:

$$
Z\cdot F\leq Y\cdot|\varepsilon|,
$$

where Z, F, and Y are defined in the theorem statement. Hence, the result is obtained:

$$
F\leq Z^{-1}\cdot Y\cdot|\varepsilon|.
$$

Since Z consists of only constant variables, i.e., it does not depend on ε, the error for each user should be smaller than some constant times the value of perturbation, ε.

Let us now check invertibility of the matrix Z by using Lemma 4. The matrix Z is strictly diagonally dominant if $\frac{1}{N-1}>\frac{C_i}{\zeta_i}$. Inserting the uniqueness condition in Theorem 1 into above condition, we find out that, if $N>\frac{\eta_i}{C_i}+1$, then the matrix Z is strictly diagonally dominant, hence nonsingular.

■

Since channel gains are not known by the users exactly, we may also encounter an estimation error in C_i. Next theorem shows that the error in the optimal transmission probabilities is a constant factor of error in C_i.

Theorem 6. *Let C_i be perturbed by a "small" scalar, ε. Then, the change in the transmission probabilities, q_i^*, is:*

$$|q_i^*(C_i+\varepsilon) - q_i^*(C_i)| \leq \frac{1}{\zeta_i}|\varepsilon|, \tag{39}$$

where $\zeta_i = k - C_i \prod_{j\neq i}(1-q_j^*)$.

Proof

$$\begin{aligned} |q_i^*(C_i+\varepsilon) - q_i^*(C_i)| &= \left| \frac{\eta_i}{k-(C_i+\varepsilon)\prod_{j\neq i}(1-q_j^*)} - \frac{\eta_i}{k-C_i\prod_{j\neq i}(1-q_j^*)} \right| \\ &= \left| \frac{\eta_i \prod_{j\neq i}(1-q_j^*)\varepsilon}{\zeta_i(\varepsilon)\zeta_i} \right| \\ &< \frac{1}{\zeta_i}|\varepsilon|, \end{aligned}$$

where $\zeta_i(\varepsilon) = k - (C_i+\varepsilon)\prod_{j\neq i}(1-q_j^*)$ and $\zeta_i = k - C_i\prod_{j\neq i}(1-q_j^*)$. Since $\zeta_i(\varepsilon)$ is greater than η_i, and $\prod_{j\neq i}(1-q_j^*)$ is smaller than one, the error in q_i^* is smaller than $\frac{1}{\zeta_i}|\varepsilon|$. Note that ζ_i only consists of constant variables. ■

Once we show that the change in transmission probabilities, q_i^*, is linear with respect to small perturbation, ε, in k and C_i, next we analyze how individual user's throughput is affected by a small error in k.

Theorem 7. *Let the pricing parameter, k, be perturbed by a "small" scalar, ε. Then, the shift in individual throughput, T_i, is:*

$$|T_i(\mathbf{q}^*(\varepsilon)) - T_i(\mathbf{q}^*)| \leq D_i|\varepsilon|, \forall i \tag{40}$$

where $D_i := C_i \sum_{j\neq i}|a_j| + C_i|a_i|$ is a positive constant and a_i is constant value obtained in Theorem 5.

Proof In Theorem 5, we have proven that transmission probabilities are linearly bounded with respect to perturbation scalar, ε, i.e., $|q_i^*(\varepsilon) - q_i^*| \leq a_i|\varepsilon|$, where a_i is some constant variable. Then, individual throughput terms are:

$$T_i(\mathbf{q}^*(\varepsilon)) = C_i^* q_i^*(\varepsilon) \prod_{j\neq i}(1 - q_j^*(\varepsilon))$$

$$T_i(\mathbf{q}^*) = C_i q_i^* \prod_{j\neq i}(1 - q_j^*) \tag{41}$$

And, the shift in individual throughput is written as:

$$T_i(\mathbf{q}^*(\varepsilon)) - T_i(\mathbf{q}^*) = C_i q_i^* \left(\prod_{j\neq i}(1 - q_j^*(\varepsilon)) - \prod_{j\neq i}(1 - q_j^*)\right) + C_i(q_i^*(\varepsilon) - q_i^*) \left(\prod_{j\neq i}(1 - q_j^*(\varepsilon)\right)$$

$$< C_i q_i^* \left(\prod_{j\neq i}(1 - q_j^*(\varepsilon)) - \prod_{j\neq i}(1 - q_j^*)\right) + C_i a_i|\varepsilon|.$$

In Theorem 5, we have shown that,

$$\left|\prod_{j\neq i}(1 - q_j^*(\varepsilon)) - \prod_{j\neq i}(1 - q_j^*)\right| \leq \left|\sum_{j\neq i} a_j|\varepsilon|\right|, \tag{42}$$

Then, it can be easily shown that

$$|T_i(\mathbf{q}^*(\varepsilon)) - T_i(\mathbf{q}^*)| \leq D_i|\varepsilon|, \tag{43}$$

where $D_i = C_i \sum_{j\neq i} |a_j| + C_i|a_i|$. ■

VII. Numerical Analysis

Our game theoretical framework and the proposed transmission probability update and dynamic pricing algorithms are analyzed numerically in MATLAB. We first investigate the rate of convergence of Algorithm 1 for varying values of step size λ_i. For this purpose, we consider a network with 50 users. The value of channel capacity is uniformly *randomly* chosen in $[0, 10]$. The maximum achievable rate is 10 bits/channel use when the signal-to-noise-ratio (SNR) is 30dB which is considered as an upper limit on SNR in the literature. A slot time is taken to be 100 microseconds. In the experiments, we observe that throughput converges in approximately 10^5 slots which corresponds to 10 seconds, when the number of users, N, is equal to 50. Thus, users wait for 10 seconds corresponding to 10^5 slots before updating their transmission probabilities. Meanwhile, the pricing parameter k takes values uniformly randomly in $[1, 20]$. Note that when $k = 20$, transmission probabilities should be smaller than 0.05, which is sufficiently

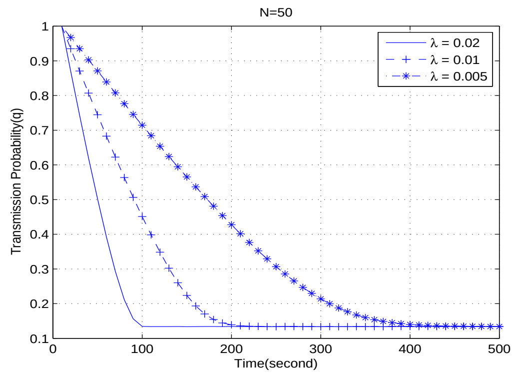

Fig. 3. Transmission probability of a user versus time for different step sizes.

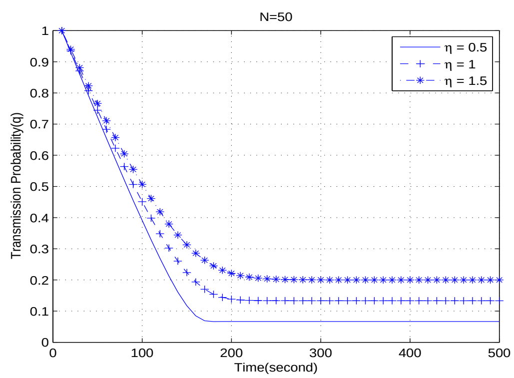

Fig. 4. Transmission probability of a user versus time for different η values.

small for our purposes. In addition, we assume that the preferences of the users towards channel access, i.e., η_i are identical, and equal to one. We have performed the simulations for three different step sizes ($\lambda = 0.02, 0.01, 0.005$), and plot the transmission probabilities for a randomly selected user in Figure 3. Note that when the step size, λ, is equal to 0.02, the convergence is faster. Here, the convergence is realized with 10 updates on transmission probabilities corresponding to 100 seconds by taking into account the time needed for convergence of the throughput. However, after convergence, the transmission probability slightly oscillates. This oscillation is due to the fact that the continuity assumption for discrete variables is violated for large step sizes. The simulation is repeated for different number of users, and the results are similar to those presented in Figure 3.

For the next scenario, we investigate the effect of the user preferences, η, on the convergence rate. We utilize the same simulation setting as in the previous case except that we vary η and we select λ as

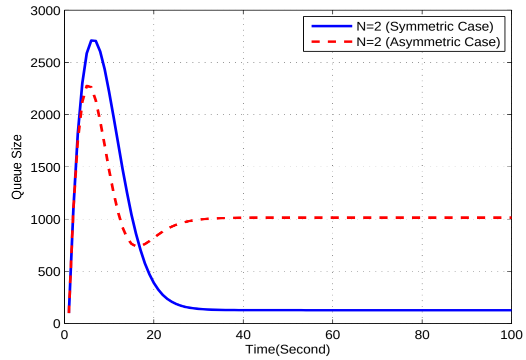

Fig. 5. Evaluation of the queue sizes in 2 node symmetric and asymmetric scenarios.

0.01. We run the simulations for different values of user preferences, e.g., $\eta = 0.5, 1.0, 1.5$. As shown in Figure 4, the convergence rate remains approximately the same for different values of η.

We next investigate the stability of user queues for the two-user symmetric and asymmetric cases. For the symmetric case, in which the system parameters are the same for all users, we select k as in (28). The capacity, C, is equal to 0.5, and the packet arrival rate is 0.1. For asymmetric case, in which users have different set of system parameter. We solve (27) numerically to obtain the value of k_i necessary to ensure the stability of user queues. The values of C_i and A_i are uniformly randomly selected in the intervals $[0, 10]$ and $[0, 1]$; however, we also ensure that the queues can be stabilized with the selected values. The step size in Algorithm 1 is selected to be $\lambda = 0.01$ for both cases. For the convergence of throughput, users wait for 1 second, since throughput converges more rapidly for small number of users. Figure 5 shows that in both cases the queue sizes first fluctuate as the algorithm converges to the equilibrium solution, $\mathbf{q}^*$. After that, they show no change confirming theoretical results.

Next, we illustrate the effect of the pricing parameter, k, on the aggregate throughput when there are 50 users with saturated queues. The value of C is selected uniformly randomly in $[0, 10]$, and we vary the value of pricing parameter, k. As shown in Fig. 6, if k is too low or too high, then the aggregate throughput decreases. When k is small, the users want to transmit with high probabilities, which results in higher number of collisions and drop rate. In the opposite extreme, when k is high, the greedy behavior of the users gets punished, and hence the channel is not fully utilized.

Finally, we investigate the aggregate throughput of the system when an adaptive *equal* pricing algorithm is implemented. In this scenario, we select the golden section search method to be implemented in our adaptive pricing algorithm. The network parameters are uniformly randomly selected as before, and the

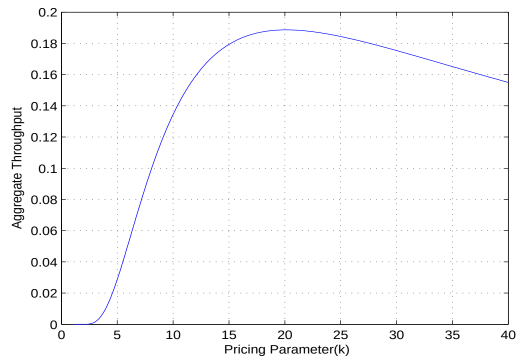

Fig. 6. Aggregate throughput versus pricing parameter k.

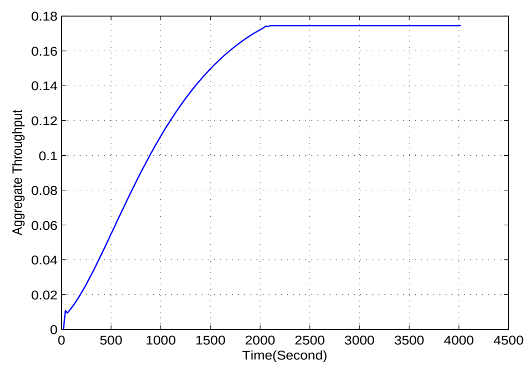

Fig. 7. Evaluation of the aggregate throughput of the system under adaptive pricing algorithm.

value of the price k is updated according to the adaptive pricing algorithm explained in Section V-B. As illustrated in Fig. 7, the algorithm successfully converges to the maximum throughput point within a few iterations. Note that, users wait for 10 seconds in order for the throughput to converge and gradient algorithm takes on the average 20 iterations for the transmission probabilities to converge. Thus, each update lasts approximately 200 seconds.

VIII. Conclusion

We have studied a noncooperative game among the users of a contention-based wireless network. The outcome of the game stabilizes the user queues and maximizes the sum of utilities or the aggregate system throughput based on the choice of pricing parameters. We have characterized the Nash equilibrium of the game and investigated the convergence properties of a distributed gradient update algorithm to compute the Nash equilibrium.

In addition, we have shown that we can move the equilibrium point to desirable regions characterized by the stability of user queues or maximization of sum of user utilities. Under full game/network knowledge, we propose an adaptive pricing algorithm that achieves social welfare, i.e., Nash equilibrium coincides with social welfare point. Under limited game/network knowledge, our adaptive pricing algorithm moves the equilibrium to the maximum throughput point.

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Appendix

We prove Theorem 3 by using Lyapunov stability, but Theorem 3 can also be proven with the method used in Theorem 8 in [13]. Our proof following Lyapunov stability is more general compared to the method in [13]. That is to say, even if the update algorithm is not gradient descent, we can prove its convergence by using the same method.

Let us introduce a candidate quadratic Lyapunov function, V, defined as,

$$V \triangleq \frac{1}{2} \sum_{i} \phi_i^2. \tag{44}$$

Note that since there is a unique equilibrium solution, $\mathbf{q}^*$, $V = 0, \forall i$ if and only if $\mathbf{q} = \mathbf{q}^*$ and $V > 0$ for all $\mathbf{q} \neq \mathbf{q}^*$, i.e., the equilibrium point corresponds to the lowest energy state of the Lyapunov function defined in (44).

Next, we need to show that Lyapunov function is a decreasing function for all values of $\mathbf{q} \neq \mathbf{q}^*$. However, we first need to calculate the first derivative of ϕ_i as:

$$\begin{aligned} \frac{\mathrm{d}\,\phi_i}{\mathrm{d}\,t} = \frac{\mathrm{d}^2\,q_i}{\mathrm{d}\,t^2} &= -\sum_{j\neq i} C_i \prod_{n\neq i,j} (1-q_n) \frac{\mathrm{d}\,q_j}{\mathrm{d}\,t} - \frac{\eta_i}{q_i^2}\frac{\mathrm{d}\,q_i}{\mathrm{d}\,t}, \\ &= -\sum_{j\neq i} \phi_j a_{ij} - \frac{\eta_i}{q_i^2}\phi_i, \end{aligned} \tag{45}$$

where $a_{ij} = C_i \prod_{n\neq i,j}(1-q_n)$.

In order for V to be a decreasing function, its first derivative should always be negative:

$$\begin{aligned} \frac{\mathrm{d}\,V}{\mathrm{d}\,t} &= \sum_i \phi_i \frac{\mathrm{d}\,\phi_i}{\mathrm{d}\,t}, \\ &= \sum_i \phi_i \left(-\sum_{j\neq i} \phi_j a_{ij} - \frac{\eta_i}{q_i^2}\phi_i \right), \qquad (46) \\ &\overset{(a)}{=} -\left(\sum_i \frac{\eta_i}{q_i^2}\phi_i^2 + \sum_i \sum_{j\neq i} a_{ij}\phi_j\phi_i \right), \\ &\overset{(b)}{=} -\left(\sum_i \frac{\sqrt{\eta_i}}{q_i}\phi_i \right)^2 - \sum_i \sum_{j>i} |\phi_i\phi_j| \left(-a_{ij} - a_{ji} + \frac{2\sqrt{\eta_i\eta_j}}{q_i q_j} \right), \\ &\overset{(c)}{\leq} -\left(\sum_i \frac{\sqrt{\eta_i}}{q_i}\phi_i \right)^2 < 0, \;\; \forall q \neq q^*, \qquad (47) \end{aligned}$$

where (a) is obtained by inserting (45) into (46), (b) follows from the square completion, and (c) follows by imposing the assumption, $a_{ij} + a_{ji} < \frac{2\sqrt{\eta_i\eta_j}}{q_i q_j}$. Note that, in (a), when $\phi_i\phi_j > 0$, $\frac{\mathrm{d}V}{\mathrm{d}t}$ is negative. Hence, in (b), we consider only the case, when $\phi_i\phi_j$ is negative.

Thus, the system is stable under the assumption of $a_{ij}+a_{ji} < \frac{2\sqrt{\eta_i\eta_j}}{q_i q_j}$. Now we investigate the conditions on C_i and k_i that realize this assumption. We claim that when $k_i + \frac{\sqrt{\eta_i\eta_j}}{2} > C_i$, then the above condition is satisfied. Furthermore, this condition is superseded by the sufficient condition for the uniqueness of the inner NE in Theorem 2.

$$\begin{aligned}
2\left(k_i + \frac{\sqrt{\eta_i\eta_j}}{2} - C_i\right)\left(k_j + \frac{\sqrt{\eta_i\eta_j}}{2} - C_j\right) &\overset{(a)}{>} 0 \\
2(k_i - C_i)(k_j - C_j) - \sqrt{\eta_i\eta_j}(C_i + C_j) + (k_i + k_j)\sqrt{\eta_i\eta_j} + \frac{\eta_i\eta_j}{2} &> 0 \\
2(k_i - C_i)(k_j - C_j) - \sqrt{\eta_i\eta_j}(C_i + C_j) &\overset{(b)}{>} 0 \\
\frac{2}{\sqrt{\eta_i\eta_j}}\left(k_i - C_i\right)\left(k_j - C_j\right) &> C_i + C_j \\
\frac{2}{\sqrt{\eta_i\eta_j}}\left(k_i - C_i\prod_{n\neq i}(1-q_n)\right)\left(k_j - C_j\prod_{m\neq j}(1-q_m)\right) &\overset{(c)}{>} \prod_{l\neq i,j}(1-q_l)(C_i + C_j) \\
\frac{2\sqrt{\eta_i\eta_j}}{q_i q_j} &\overset{(d)}{>} a_{ij} + a_{ji}.
\end{aligned}$$

where (a) follows from the condition, $k_i + \frac{\sqrt{\eta_i\eta_j}}{2} - C_i > 0$, (b) follows from $(k_i + k_j)\sqrt{\eta_i\eta_j} + \frac{\eta_i\eta_j}{2} > 0$ and (c) follows from $k_i - C_i\prod_{n\neq i}(1-q_n) > k_i - C_i$ and $\prod_{l\neq i,j}(1-q_l)(C_i + C_j) < C_i + C_j$. In (d), we use the definitions of q_i and a_{ij}. Therefore, the gradient update algorithm converges asymptotically to the unique NE of the game.

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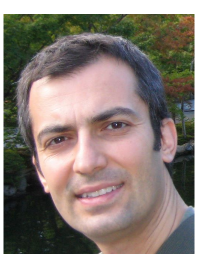

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