Distributed and Robust Rate Control for Communication Networks

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Abstract Contemporary networks are distributed, complex, and heterogeneous. Ensuring an efficient, fair, and incentive-compatible allocation of bandwidth among their users constitutes a challenging and multi-faceted research problem. This chapter presents three control and game-theoretic approaches that address rate control problems from different perspectives. First, a noncooperative rate control game focusing on incentive compatibility issues is formulated. Secondly, a primal-dual algorithm incorporating queue dynamics and maximizing a global objective is considered. Finally, a robust rate control framework is presented. For each scheme, the respective equilibrium, stability, and robustness properties are rigorously analyzed and discussed.

1 Introduction

Wired and wireless communication networks are an ubiquitous and indispensable part of the modern society. They serve a variety of purposes and applications for their end users. Hence, networks exhibit heterogeneous characteristics in terms of their access infrastructure (e.g. wired vs. wireless), protocols, and capacity. Moreover, contemporary networks such as the Internet are heavily decentralized both in their administration and resources.

The end users of communication networks are also diverse and run a variety of applications ranging from multimedia (VoIP, video) to gaming and data communications. As a result of the networks' distributed nature, users often have little information about the network topology and characteristics. Regardless, they can behave selfishly in their demands for bandwidth.

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Given the mentioned characteristics of the contemporary networks and their users, a fundamental research question is: how to ensure efficient, fair, and incentive-compatible allocation of network bandwidth among its users. Complicating the problem further, the mentioned objectives have to be achieved through distributed algorithms while ensuring robustness with respect to information delays and capacity changes. This research challenge can be quite open-ended due to the multi-faceted nature of the underlying problems.

1.1 Summary and Contributions

This chapter presents three control and game-theoretic approaches that address the described rate control problem from different perspectives. The objective here is to investigate the underlying mathematical principles of the problem and solution concepts rather than discussing possible implementation scenarios. However, it is hoped that the rigorous mathematical analysis presented will be useful as a basis for engineering future rate control schemes.

A noncooperative rate control game is presented in Section 3. Adopting a utility-based approach, user preferences are captured by a fairly general class of cost functions [4]. Based on their own utility functions and external prices, the users (players) of this game use a standard gradient algorithm to update their flow rates iteratively over time, resulting in an end-to-end congestion control scheme. The game admits a unique Nash equilibrium under a sufficient condition, where no user has an incentive to deviate from it. Furthermore, a mild symmetricity assumption and a sufficient condition on maximum delay ensure its global stability with respect to the gradient algorithm for general network topologies and under fixed heterogeneous delays. The upper bound on communication delays given in the sufficient condition is inversely proportional to the square root of the number of users sharing a link multiplied by the cube of a gain constant.

Section 4 studies a primal-dual rate control scheme to solve a global optimization problem, where each user's cost function is composed of a pricing function proportional to the queueing delay experienced by the user, and a fairly general utility function which captures the user's demand for bandwidth [5]. The global objective is to maximize the sum of user utilities under fixed capacity constraints. Using a network model based on fluid approximations and through a realistic modeling of queues, the existence of a unique equilibrium is established, at which the global optimum is achieved. The scheme is globally asymptotically stable for a general network topology. Furthermore, sufficient conditions for system stability are derived when there is a bottleneck link shared by multiple users experiencing non-negligible communication delays.

A robust flow control framework is introduced in Section 5. It is based on an H^∞-optimal control formulation for allocating rates to devices on a network with heterogeneous time-varying characteristics [6]. H^∞ methods are used in control theory to synthesize controllers achieving robust performance or stabilization. Here,

H^∞ analysis and design allow for the coupling between different devices to be relaxed by treating the dynamics for each device independently from others. Thus, the resulting distributed end-to-end rate control scheme relies on minimum information and achieves fair and robust rate allocation for the devices. In the fixed capacity case, it is shown that the equilibrium point of the system ensures full capacity usage by the users.

The formulations presented in Sections 3, 4, and 5 are, on the one hand, closely related to each other. Each approach mainly shares the same common network model, which will be discussed in Section 2. Furthermore, they are totally distributed, end-to-end schemes with little information exchange overhead. All of the schemes are robust with respect to information delays in the system and their stability properties are analyzed rigorously.

On the other hand, each approach brings the problem of rate allocation a different perspective. The rate control game of Section 3 focuses mainly on incentive compatibility and adopts Nash equilibrium as the preferred solution concept. The primal-dual scheme of Section 4 extends the basic fluid network model by taking into account the queue dynamics and is built upon available information to users for decision making. The robust rate control scheme of Section 5 emphasizes robustness with respect to capacity changes and delays. It also differs from the previous two, which share a utility-based approach, by focusing on fully efficient usage of the network capacity.

The remainder of the chapter is organized as follows. A brief overview of existing relevant literature is discussed next. The network model and its underlying assumptions are presented in Section 2. Section 3 studies a rate control game along with an equilibrium and stability analysis under information delays. In Section 4, a primal-dual scheme is investigated. Section 5 presents a robust control framework and its analysis. The chapter concludes with remarks in Section 6.

1.2 Related Work

The research community has intensively studied the challenging problem of rate and congestion control in the recent years. Consequently, a rich literature, which investigates the problem using a variety of approaches, has emerged. While it is far from a comprehensive survey, a small subset of the existing body of literature on the subject is summarized here as a reference.

After the introduction of the congestion control algorithm for Transfer Control Protocol (TCP) [18], research community has focused on modeling and analysis of rate control algorithms. Based on an earlier work by Kelly [21], Kelly, Mauloo, and Tan [20] have presented the first comprehensive mathematical model, and posed the underlying resource allocation problem as one of constrained optimization. The primal and dual algorithms that they have introduced are based on user utility and link pricing (explicit congestion feedback) functions, where the sum of user utilities are maximized within the capacity (bandwidth) constraints of the links. They have also introduced the concept of proportional fairness, which is a relaxed version of min-max fairness [31], as a resource allocation criterion among users.

Subsequent studies [14, 23, 24, 26, 29] have investigated variations and generalizations of the distributed congestion control framework of [20, 21]. Low and Lapsley [26] have analyzed the convergence of distributed synchronous and asynchronous discrete algorithms, which solve a similar optimization problem. Mo and Walrand [29] have generalized the proportional fairness, and have proposed a fair end-to-end window-based congestion control scheme, which is similar to the primal algorithm. The main difference of this window-based algorithm from the primal algorithm is that it does not need explicit congestion feedback from the routers. Instead it makes use of measured queuing delay as implicit congestion feedback. La and Anantharam [24] have considered a system model similar to proposed in [29] with a window-based control scheme and static modeling of link buffers. They have investigated convergence properties of the proposed charge-sensitive congestion control scheme, which utilizes a static pricing scheme based on link queueing delays. In addition, they have established stability of the algorithm at a single bottleneck node. Kunniyur and Srikant [23] have examined the question of how to provide congestion feedback from the network to the user. They have proposed an explicit congestion notification (ECN) marking scheme combined with dynamic adaptive virtual queues, and have shown using a time-scale decomposition that the system is semi-globally stable in the no-delay case.

In developing rate control mechanisms for the Internet, game theory provides a natural framework. Users on the network can be modeled as players in a rate control game where they choose their strategies, or in this case, flow rates. Players are selfish in terms of their demands for network resources, and have no specific information on other users' strategies. A user's demand for bandwidth is captured in a utility function which may not be bounded. To compensate for this, one can devise a *pricing* function, proportional to the bandwidth usage of a user, as a disincentive to him to have excessive demand for bandwidth. This way, the network resources are preserved, and an incentive is provided for the user to implement end-to-end congestion control [16]. A useful concept in such a noncooperative congestion control game is Nash equilibrium [10] where each player minimizes its own cost (or maximizes payoff) given all other players' strategies. There is, consequently, a rich literature on game theoretic analysis of flow control problems utilizing both cooperative [34] and noncooperative [1–3, 5, 7, 8, 30] frameworks.

Robustness of distributed rate control algorithms with respect to delays in the network have been investigated by many studies [19, 27, 32]. Johari and Tan [19] have analyzed the local stability of a delayed system where the end user implements the primal algorithm. They have considered a single link accessed by a single user, as well as its multiple user extension under the assumption of symmetric delays. In both cases, they have provided sufficient conditions for local stability of the underlying system of equations. Massoulie [27] has extended these local stability results to general network topologies and heterogeneous delays. In another study, Vinnicombe [32] has also provided sufficient conditions for local stability of a user rate control scheme which is a generalization of the same algorithm. Elwalid [15] has considered stability of a linear class of algorithms where the source rate varies in proportion to the difference between the buffer content and the target value. Deb and Srikant [14], on the other hand, have focused on the case of single user and a single resource and investigated sufficient conditions for global stability of various nonlinear congestion control schemes under fixed information delays. Liu, Başar, and Srikant [25] have extended the framework of [20, 21] by introducing a primal-dual algorithm which has dynamic adaptations at both ends (users and links), and have given a condition for its local stability under delay using the generalized Nyquist criterion. Wen and Arcak [33] have used a passivity framework to unify some of the stability results on primal and dual algorithms without delay, have introduced and analyzed a larger class of such algorithms for stability, and have shown robustness to variations due to delay.

2 Network Model

A general network model is considered which is based on fluid approximations. Fluid models are widely used in addressing a variety of network control problems, such as congestion control [1, 7, 29], routing [7, 30], and pricing [11, 20, 34]. The topology of the network is characterized by a connected graph consisting of a set of nodes $\mathscr{N} = \{1,\dots,N\}$ and a set of links $\mathscr{L} = \{1,\dots,L\}$. Each link $l \in \mathscr{L}$ has a fixed capacity $C_l > 0$, and is associated with a buffer of size $b_l \geq 0$.

There are M active users sharing the network, $\mathscr{M} = \{1,\dots,M\}$. For simplicity, each user is associated with a (unique) connection. Hence, the i^{th} $(i \in \mathscr{M})$ user corresponds to a unique connection between a source and a destination node, $s_i, de_i \in \mathscr{N}$. The i^{th} user sends its nonnegative flow, $x_i \geq 0$, over its route (path) R_i, which is a subset of $\mathscr{L}$. An upper bound, $x_{i,max}$, is imposed on the i^{th} users flow rate, which may be due to a user (device) specific physical limitation.

Define a routing matrix, $\mathbf{A} := [(a_{l,i})]$ of ones and zeros, as in [20] which describes the relation between the set of routes $\mathscr{R} = \{1,\dots,M\}$ associated with the users (connections), $i \in \mathscr{M}$, and links $l \in \mathscr{L}$:

$$A_{l,i} = \begin{cases} 1, & \text{if source } i \text{ uses link } l \\ 0, & \text{if source } i \text{ does not use link } l \end{cases} . \tag{1}$$

It is assumed here, without any loss of generality, that no rows or columns in $\mathbf{A}$ are identically zero. Using this routing matrix $\mathbf{A}$, the capacity constraints of the links are given by

$$\mathbf{Ax} \leq \mathbf{C},$$

where $\mathbf{x}$ is the $(M \times 1)$ flow rate vector of the users and $\mathbf{C}$ is the $(L \times 1)$ link capacity vector. The flow rate vector, $\mathbf{x}$, is said to be feasible if it is nonnegative and satisfies this constraint. Let $\mathbf{x_{-i}}$ be the flow rate vector of all users except the i^{th} one. For a given fixed, feasible $\mathbf{x_{-i}}$, there exists a strict finite upper-bound $m_i(\mathbf{x_{-i}})$ on flow rate of the i^{th} user, x_i, based on the capacity constraints of the links:

$$m_i(\mathbf{x_{-i}}) = \min_{l \in R_i}(c_l - \sum_{j \neq i} A_{l,j}\, x_j) \;\geq\; 0\,.$$

2.1 Model Assumptions

Simplifying assumptions are necessary to develop a mathematically tractable model. The assumptions of this chapter are shared by the majority of the literature on the subject, including the works cited in Section 1. Furthermore, the analytical results obtained based on these assumptions are verified many times via realistic packet level simulations in the literature. The main assumptions on the network model are summarized as:

1. The network model is based on fluid approximations, where individual packets are replaced with flows. Fluid models are widely used in addressing a variety of network control problems, such as congestion control [1, 7, 29], routing [7, 30], and pricing [11, 20, 34].
2. For simplicity, each user is associated with a unique connection and a corresponding fixed route (path). The routing matrix **A** is assumed to be of full row rank as non-bottleneck links have no effect on the equilibrium point due to zero queuing delay on those links.
3. Bandwidth is focused on as the main network resource.
4. Information delays are assumed to be fixed for tractability of analysis.
5. The links are associated with first-in first-out (FIFO) finite queues (buffers) with droptail packet dropping policies.

Additional assumptions, being part of the specific optimization or game formulations, are explicitly introduced and discussed in their respective sections.

3 Rate Control Game

A noncooperative rate control game is played among M users on the general network model, which is described in the previous section. The game is noncooperative as the users are assumed to be selfish in terms of their demand for bandwidth and have no means of communicating with each other about their preferences. Hence, each user tries to optimize his usage of the network independently by minimizing its own specific cost function J_i. This cost function is defined on the compact, continuous set of feasible rates of users, $\mathbf{x} := \{\mathbf{x} \subset \mathbb{R}^M : \mathbf{x} \geq 0, \mathbf{Ax} \leq \mathbf{C}\}$. The cost function J_i not only models the user preferences but it also includes a feedback term capturing the current network state. Thus, the i^{th} user minimizes his cost, J_i, by adjusting his flow rate $0 \leq x_i \leq m_i(\mathbf{x_{-i}})$ given the fixed, feasible flow rates of all other users on its path, $\{x_j : j \in (R_j \cap R_i)\}$.

The cost function of the i^{th} user, J_i, is defined as the difference between a user-specific pricing function, P_i, and a utility function, U_i. It is smooth, i.e. at least twice continuously differentiable in all its arguments. The pricing function P_i depends on the current state of the network, and can be interpreted as the price a user pays for using the network resources. The utility function U_i is defined to be increasing and concave in accordance with elastic traffic as well as with the economic principle, law of diminishing returns. The utility of each user depends only on its own flow rate. Thus, the cost function is defined as

$$J_i(\mathbf{x};\mathbf{C},\mathbf{A}) = P_i(\mathbf{x};\mathbf{C},\mathbf{A}) - U_i(x_i). \tag{2}$$

Here, the pricing function P_i of user i does not necessarily depend on the flow rates of *all* other users; it can be structured to depend only on the flow rates of the users sharing the same links on the path of the i^{th} user.

The rate control game defined proposes "pricing" as a way to enforce a more favorable outcome for the system and users. If there is no pricing scheme, then the increasing and concave user utilities result in a solution where each user sends with the maximum possible rate. In practice, this would lead to a congestion collapse or "tragedy of commons". To remedy this, the function P, proportional to the bandwidth usage of a user, is utilized as an incentive for the user to curb excessive demand. Thus, the network resources are preserved, and an incentive is provided for the user to implement end-to-end congestion control. On the other hand, the analysis in this section focuses on mathematical principles rather than architectural concerns or possible implementation of such pricing schemes.

3.1 Nash Equilibrium as a Unique Solution

The defined rate control game may admit a (unique) Nash equilibrium (NE) as a solution. In this context, Nash equilibrium is defined as a set of flow rates, $\mathbf{x}^*$ (and corresponding costs J^*), with the property that no user can benefit by modifying its flow while the other players keep their flows fixed. Furthermore, if the Nash equilibrium, $\mathbf{x}^*$, meets the capacity constraints (e.g. $\mathbf{A}\mathbf{x}^* \leq \mathbf{C}$) as well as the positivity constraint ($\mathbf{x}^* \geq 0$) with strict inequality, then it is an *inner* solution.

Definition 1 (Nash Equilibrium). The user flow rate vector $\mathbf{x}^*$ is in Nash Equilibrium, when x_i^* of any i^{th} user is the solution to the following optimization problem given that all users on its path have equilibrium flow rates, $\mathbf{x}^*_{-i}$:

$$\min_{0 \leq x_i \leq m_i(\mathbf{x_{-i}}^*)} J_i(x_i, x^*_{-i}, \mathbf{C}, \mathbf{A}) \;, \tag{3}$$

where $\mathbf{x_{-i}}$ denotes the collection $\{x_j : j \in R_j \cap R_i\}_{j=1,\dots,M}$.

The assumptions on the user cost functions are next formalized:

A1. $P_i(\mathbf{x})$ is jointly continuous in all its arguments and twice continuously differentiable, non-decreasing and convex in x_i, i.e.

$$\frac{\partial P_i(\mathbf{x})}{\partial x_i} \geq 0, \quad \frac{\partial^2 P_i(\mathbf{x})}{\partial x_i^2} \geq 0. \tag{4}$$

A2. $U(x_i)$ is jointly continuous in all its arguments and twice continuously differentiable, non-decreasing and strictly concave in x_i, i.e.

$$\frac{\partial U_i(x_i)}{\partial x_i} \geq 0, \quad \frac{\partial^2 U_i(x_i)}{\partial x_i^2} < 0 \ \ , \forall x_i$$

Moreover, the optimal solution (Nash equilibrium) is an *inner* one, $0 < \sum_j A_{l,j} x_j^* < c_l \ , \forall l$, under the additional assumption:

A3. The i^{th} user's cost function has the following properties at $x_i = 0$:

$$\frac{\partial J_i(\mathbf{x} : x_i = 0)}{\partial x_i} < 0, \ \forall \mathbf{x},$$

and at $x_i = m_i(\mathbf{x_{-i}})$,

$$\frac{\partial J_i(\mathbf{x} : x_i = m_i(\mathbf{x_{-i}}))}{\partial x_i} > 0, \ \forall \mathbf{x}.$$

Theorem 1 establishes that the congestion control game admits a unique NE under the following further assumption:

A4. The price function $P_i(\mathbf{x})$ of the i^{th} user is defined as the sum of link price functions on its path,

$$P_i = \sum_{l \in R_i} P_l(\sum_{j: l \in R_j} x_j),$$

where P_l is defined as a function of the aggregate flow on link l, and satisfies (4) with i replaced by l.

Theorem 1. *Under A1-A4, the network game admits a unique inner Nash equilibrium.*

Proof. Earlier versions of this proof are in [3] and [4]. Let $X := \{\mathbf{x} \in \mathbb{R}^M : \mathbf{Ax} \leq \mathbf{C} \ , \mathbf{x} \geq 0\}$ be the set of feasible flow rate vectors (or strategy space) of the users. The flow rate of a generic i^{th} user is nonnegative and bounded above by the minimum link capacity on its route, $0 \leq x_i < \min_{l \in R_i} C_l$. The set X is clearly closed and bounded, hence, compact. First, X is shown to have a nonempty interior and be convex. Define the following flow rate vector: $\mathbf{x}^{max} := \frac{1}{M} \min_l C_l$. Clearly, $\mathbf{x}^{max} \in X$ is feasible and positive as $C_l > 0, \forall l$. Hence, there exists at least one positive and feasible flow rate vector in the set X, which is an interior point. Thus, the set X has a nonempty interior. Let $\mathbf{x}^1$, $\mathbf{x}^2 \in X$ be two feasible flow rate vectors, and $0 < \lambda < 1$ be a real number. For any $\mathbf{x}^\lambda := \lambda \mathbf{x}^1 + (1 - \lambda)\mathbf{x}^2$, it follows that

$$\mathbf{Ax}^\lambda = \mathbf{A}(\lambda \mathbf{x}^1 + (1 - \lambda)\mathbf{x}^2) \leq \mathbf{C}$$

Furthermore, $\mathbf{x}^{\lambda} \geq 0$ by definition. Hence, $\mathbf{x}^{\lambda}$ is feasible and is in X for any $0 < \lambda < 1$. Thus, the set X is convex. By a standard theorem of game theory (Thm. 4.4 p.176 in [10]), the network game admits a NE.

Next, uniqueness of the NE is shown. Differentiating (2) with respect to x_i, and using assumptions *A1* and *A2* results in

$$f_i(\mathbf{x}) := \frac{\partial J_i(\mathbf{x})}{\partial x_i} = \frac{\partial P_i(\mathbf{x})}{\partial x_i} - \frac{\partial U_i(x_i)}{\partial x_i} \ . \tag{5}$$

As a simplification of notation, $\mathbf{C}$ and $\mathbf{A}$ are suppressed as arguments of the functions for the rest of this proof.

Differentiating $J_i(\mathbf{x})$ twice with respect to x_i yields

$$\frac{\partial f_i(\mathbf{x})}{\partial x_i} = \frac{\partial^2 J_i(\mathbf{x})}{\partial x_i^2} = \frac{\partial^2 P_i(\mathbf{x})}{\partial x_i^2} - \frac{\partial^2 U_i(x_i)}{\partial x_i^2} > 0.$$

Hence, J_i is unimodal and has a unique minimum. Based on $A3$, $f_i(\mathbf{x})$ attains the zero value at $m_i(\mathbf{x_{-i}}) > x_i > 0$ given a fixed feasible $\mathbf{x_{-i}}$. Thus, the optimization problem (3) admits a unique positive solution.

To preserve notation, let $\frac{\partial^2 J(\mathbf{x})}{\partial x_i^2}$ be denoted by B_i. Further introduce, for $i, j \in \mathscr{M}$, $j \neq i$,

$$\frac{\partial^2 J_i(\mathbf{x})}{\partial x_i \partial x_j} = \frac{\partial^2 P_i(\mathbf{x})}{\partial x_i \partial x_j} =: A_{i,j} \ ,$$

with both B_i and $A_{i,j}$ defined on the space where $\mathbf{x}$ is nonnegative, and bounded by the link capacities. Suppose that there are two Nash equilibria, represented by two flow vectors $\mathbf{x}^0$ and $\mathbf{x}^1$, with elements x_i^0 and x_i^1, respectively. Define the pseudo-gradient vector:

$$g(\mathbf{x}) := \left[\nabla_{x_1} J_1(\mathbf{x})^T \cdots \nabla_{x_M} J_M(\mathbf{x})^T\right]^T \tag{6}$$

As the Nash equilibrium is necessarily an inner solution, it follows from first-order optimality condition that $g(\mathbf{x}^0) = 0$ and $g(\mathbf{x}^1) = 0$. Define the flow vector $\mathbf{x}(\theta)$ as a convex combination of the two equilibrium points $\mathbf{x}^0$ and $\mathbf{x}^1$:

$$\mathbf{x}(\theta) = \theta \mathbf{x}^0 + (1-\theta)\mathbf{x}^1$$

where $0 < \theta < 1$. By differentiating $\mathbf{x}(\theta)$ with respect to θ,

$$\frac{dg(\mathbf{x}(\theta))}{d\theta} = G(\mathbf{x}(\theta)) \frac{dx(\theta)}{d\theta} = G(\mathbf{x}(\theta))(\mathbf{x}^1 - \mathbf{x}^0) \ , \tag{7}$$

where $G(\mathbf{x})$ is the Jacobian of $g(\mathbf{x})$ with respect to $\mathbf{x}$:

$$G(\mathbf{x}) := \begin{pmatrix} B_1 & A_{12} & \cdots & A_{1M} \\ \vdots & & \ddots & \vdots \\ A_{M1} & A_{M2} & \cdots & B_M \end{pmatrix}_{M \times M} . \tag{8}$$

Additionally note that, by assumption *A4*:

$$\sum_{l\in(R_i\cap R_j)} \frac{\partial^2 J_l(\mathbf{x})}{\partial x_i \partial x_j} = \sum_{l\in(R_i\cap R_j)} \frac{\partial^2 J_l(\mathbf{x})}{\partial x_i \partial x_j}$$

$$\Rightarrow A(i,j) = A(j,i) \;\; i,j \in \mathscr{M} \; .$$

Hence, $G(\mathbf{x})$ is symmetric. Integrating (7) over θ,

$$0 = g(\mathbf{x}^1) - g(\mathbf{x}^0) = \left[\int_0^1 G(\mathbf{x}(\theta))d\theta\right](\mathbf{x}^1 - \mathbf{x}^0)\;, \tag{9}$$

where $(\mathbf{x}^1 - \mathbf{x}^0)$ is a constant flow vector. Let $\overline{B_i(\mathbf{x})} = \int_0^1 B_i(\mathbf{x}(\theta))d\theta$ and $\overline{A_{ij}(\mathbf{x})} = \int_0^1 A_{ij}(\mathbf{x}(\theta))d\theta$. In view of *A2* and *A4*, $B_i(\mathbf{x}) > A_{ij}(\mathbf{x}) > 0$, $\forall i,j$. Thus, $\overline{B_i(\mathbf{x})} > \overline{A_{ij}(\mathbf{x})} > 0$, for any $\mathbf{x}(\theta)$. In order to simplify the notation, define the matrix $\mathscr{G}(\mathbf{x}^1,\mathbf{x}^0) := \int_0^1 G(\mathbf{x}(\theta))d\theta$, which can be shown to be full rank for any fixed $\mathbf{x}$. Rewriting (9) as, $0 = \mathscr{G}\cdot[\mathbf{x}^1 - \mathbf{x}^0]$, since $\mathscr{G}$ is full rank, it readily follows that $\mathbf{x}^1 - \mathbf{x}^0 = 0$. Therefore, the NE is unique.

Under *A3*, the NE has to be an inner solution, as the following argument shows. First, $\mathbf{x} \geq 0$, with $x_i = 0$ for at least one i, cannot be an equilibrium point since user i can decrease its cost by increasing its flow rate. Similarly, the boundary points $\{\mathbf{x} \in \mathbb{R}^M : \mathbf{A}\mathbf{x} \leq \mathbf{C}\;, \mathbf{x} \geq 0, \text{with } (Ax)_l = C_l \text{ for at least one link } l\}$ cannot constitute NE, as users whose flows pass through the link can decrease their flow rates under *A3*. Thus, under *A1-A4*, the network game admits a unique inner NE.

3.2 Stability Analysis

Consider a simple dynamic model of the defined rate control game where each user changes his flow rate in proportion with the gradient of his cost function with respect to his flow rate. Note that this corresponds to the well-known steepest descent algorithm in nonlinear programming [12]. Hence, the user update algorithm is:

$$\dot{x}_i(t) = \frac{dx_i(t)}{dt} = -\frac{\partial J_i(\mathbf{x}(t))}{\partial x_i} = \frac{dU_i(x_i)}{dx_i} - \sum_{l\in R_i} f_l\left(\sum_{j\in\mathscr{M}_l} x_j\right) := \theta_i(\mathbf{x}), \tag{10}$$

for all $i = 1,\ldots,M$, where $\mathscr{M}_l(M_l)$ is the set (number) of users whose flows pass through the link, $l \in R_i$; t is the time variable, which we drop in the second line for a more compact notation; and f_l is defined as $f_l(\cdot) := \partial P_l(\cdot)/\partial x_i$.

By assumption *A4*, the partial derivative of f_l with respect to x_i, $\partial f_l(\cdot)/\partial x_i$, is non-negative. Furthermore, since $P_l(\mathbf{x})$ is convex and jointly continuous in x_i for all i whose flows pass through the link l, on the compact set of feasible flow rate vectors, $X := \{\mathbf{x} \in \mathbb{R}^M : \mathbf{A}\mathbf{x} \leq \mathbf{C}\;, \mathbf{x} \geq 0\}$, the derivative $\partial f_l(.)/\partial x_i$ can be bounded above by a constant $\alpha_l > 0$. Hence,

$$0 \leq \frac{\partial f_l(\bar{x}_l)}{\partial x_i} \leq \alpha_l, \tag{11}$$

where $\bar{x}_l = \sum_{i \in \mathscr{M}_l} x_i$.

It is now shown that the system defined by (10) is asymptotically stable on the set X, which is invariant by assumption *A3* under the gradient update algorithm (10). In order to see the invariance of X, each boundary of X is investigated separately. If $x_i = 0$ for some $i \in \mathscr{M}$, then $\dot{x}_i > 0$ follows from (10) under assumption *A3* due to the gradient descent algorithm of user i. Hence, the system trajectory moves toward inside of X. Likewise, in the case of $\bar{x}_l = C_l$ for some $l \in \mathscr{L}$, it follows from (10) and assumption *A3* that $\dot{x}_i < 0 \ \forall i \in \mathscr{M}_l$, and hence, the trajectory remains inside the set X.

The unique inner NE, $\mathbf{x}^*$, (see Theorem 1) of the rate control game constitutes the equilibrium state of the dynamical system (10) in X. Around this equilibrium, define a candidate Lyapunov function $V : \mathbb{R}^M \to \mathbb{R}^+$ as

$$V(\mathbf{x}) := \frac{1}{2} \sum_{i=1}^{M} \theta_i^2(\mathbf{x}),$$

which is in fact restricted to the domain X. Further let $\Theta := [\theta_1, \ldots, \theta_M]$. Taking the derivative of V with respect to t on the trajectories generated by (10), one obtains

$$\dot{V}(\mathbf{x}) = \sum_{i=1}^{M} \frac{d^2 U_i(x_i)}{dx_i^2} \theta_i^2(\mathbf{x}) - \Theta^T(\mathbf{x}) A^T K A \Theta(\mathbf{x}),$$

where A is the routing matrix, and K is a diagonal matrix defined as

$$K := \mathrm{diag}\Big[\frac{\partial \mathrm{f}_1(\bar{\mathrm{x}})}{\partial \bar{\mathrm{x}}}, \frac{\partial \mathrm{f}_2(\bar{\mathrm{x}})}{\partial \bar{\mathrm{x}}}, \ldots, \frac{\partial \mathrm{f}_\mathrm{M}(\bar{\mathrm{x}})}{\partial \bar{\mathrm{x}}}\Big].$$

Since $A^T K A$ is non-negative definite and $d^2 U_i / dx_i^2$ is uniformly negative definite, $V(\mathbf{x})$ is strictly decreasing, $\dot{V}(\mathbf{x}) < 0$, on the trajectory of (10). Thus, the system is asymptotically stable on the invariant set X by Lyapunov's stability theorem (see Theorem 3.1 in [22]).

Theorem 2. *Assume A1-A4 hold. Then, the unique inner Nash equilibrium of the network game is globally stable on the compact set of feasible flow rate vectors,* $X := \{\mathbf{x} \in \mathbb{R}^M : \mathbf{A}\mathbf{x} \leq \mathbf{C}\ , \mathbf{x} \geq 0\}$ *under the gradient algorithm given by*

$$\dot{x}_i = -\frac{\partial J_i(\mathbf{x})}{\partial x_i}\ , i = 1, \ldots, M.$$

3.3 Stability under Information Delays

Whether the user rate control (gradient) algorithm (10) is robust with respect to communication delays is an important question. This section investigates the rate control scheme under bounded and heterogeneous communication delays. The distributed rate control algorithm under communication delays is defined as

$$\dot{x}_i(t) = \frac{dU_i(x_i(t))}{dx_i} - \sum_{l \in R_i} f_l\left(\sum_{j \in \mathscr{M}_l} x_j(t - r_{li} - r_{lj}) \right) \tag{12}$$

where r_{li} and r_{lj} are fixed communication delays between the l^{th} link and the i^{th} and j^{th} users respectively. It is implicitly assumed here that queueing delays are negligible compared to the fixed propagation delays in the system.

3.3.1 Notation and Definitions

Notice again that, f_l is defined as $f_l(\cdot) := \partial P_l(\cdot)/\partial x_i$ and the pricing function of the i^{th} user is defined in accordance with assumption *A4* as

$$P_i = \sum_{l \in R_i} P_l\big(\sum_{j \in \mathscr{M}_l} x_j \big),$$

where R_i is the path (route) of user i, and P_l is the pricing function at link $l \in \mathscr{L}$. The notation is simplified by defining

$$\bar{x}_l^i(t-r) := \sum_{j \in \mathscr{M}_l} x_j(t - r_{li} - r_{lj}).$$

In addition, let q be an upper-bound on the maximum round-trip time (RTT) in the system:

$$q := 2 \max_i \sum_{l \in R_i} r_{li} - r_{(l-1)i},$$

where $r_{0i} = 0 \; \forall i$. Finally, define $\mathbf{x}_t := \{\mathbf{x}(t+s),\; -q \le s \le 0\}$, and by a slight abuse of notation let $\theta_i(\mathbf{x}_t)$ denote the right hand side of (12).

Let $\phi_i \in C\big([-r_i, 0], \mathbb{R}\big)$ be a feasible flow rate function (initial condition) for the i^{th} user's dynamics (12) at time $t = 0$, where C is the set of continuous functions. In addition, let $\mathbf{x}(\phi)(t)$ be the solution of (12) through ϕ for $t \ge 0$, and $\dot{\mathbf{x}}(\phi_i)(t)$ be its derivative. In order to simplify the notation, $\mathbf{x}(\phi)$ and $\mathbf{x}$ as well as $\theta(\phi)$ and θ and their respective derivatives will be used interchangeably for the remainder of the paper.

Finally, a continuously differentiable and positive function $V : C^M \to \mathbb{R}^+$ is defined as

$$V(\mathbf{x}_t(\phi)) := \frac{1}{2} \sum_{i=1}^{M} \theta_i^2(\mathbf{x}_t(\phi)) = \frac{1}{2} \Theta^T(\mathbf{x}_t(\phi)) \Theta(\mathbf{x}_t(\phi)).$$

This constitutes the basis for the following candidate *Lyapunov functional* $\overline{V} : \mathbb{R}^+ \times C^M \to \mathbb{R}^+$,

$$\overline{V}(t;\phi) := \sup_{t-2q \le s \le t} V(\mathbf{x}_s(\phi)),$$

where $V(\mathbf{x}_s) = 0,\ s \in [-2q, -q]$ without any loss of generality.

3.3.2 Stability Analysis under Delays

In order to establish global stability under delays, it is shown that the Lyapunov functional $\overline{V}(t;\phi)$ is non-increasing. Furthermore, the stability theory for autonomous systems of [17] is utilized to generalize the scalar analysis of [35] and also of Chapter 5.4 of [17] to the multidimensional (multi-user) case. Let $\dot{\overline{V}}(t;\phi)$ and $\dot{V}(t;\phi)$ be defined as the upper right-hand derivatives of $\overline{V}(t;\phi)$ and $V(t;\phi)$ respectively along $\mathbf{x}_t(\phi)$. In order for $\overline{V}(t;\phi)$ to be non-increasing, i.e. $\dot{\overline{V}}(t;\phi) \le 0$, the set

$$\Phi = \{\phi \in C : \overline{V}(t;\phi) = V(\mathbf{x}_t(\phi)); \dot{V}(\mathbf{x}_t(\phi)) > 0\ \forall t \ge 0\} \tag{13}$$

has to be empty. This is established in the following lemma.

Lemma 1. *The set Φ, defined in (13) is empty if the following condition is satisfied*

$$q \le \frac{2\sqrt{\bar{d}}}{\sqrt{M} b^{3/2}},$$

where

$$b := \max_i \left(-\min_{x_i \in X} \frac{d^2 U_i(x_i)}{dx_i^2} + \sum_{l \in R_i} M_l \alpha_l \right),$$

and

$$\bar{d} := \min_i \min_{x_i \in X} \left| \frac{d^2 U_i(x_i)}{dx_i^2} \right|.$$

Proof. To see this consider the case when the set Φ is not empty. Then, by definition, there exists a time t and an $h > 0$ such that $\dot{\overline{V}}(\mathbf{x}_{t+h}(\phi)) > \dot{\overline{V}}(\mathbf{x}_t(\phi))$, and hence, $\dot{V}(\mathbf{x}_t(\phi))$ cannot be non-increasing.

It is now shown that the set Φ is indeed empty. Assume otherwise. Then, for any given t, there exists an $\varepsilon > 0$ such that

$$\overline{V}(t;\phi) = V(\mathbf{x}_t(\phi)) = \sum_{i=1}^{M} \theta_i^2(\mathbf{x}_t(\phi)) = \varepsilon \tag{14}$$

and

$$V(\mathbf{x}_s(\phi)) = \sum_{i=1}^{M} \theta_i^2(\mathbf{x}_s) \le \varepsilon\ ,\ s \in [t-2q\,,t].$$

Thus, the following bound on θ_i, and thus on $\dot{x}_i$, follows immediately:

$$|\theta_i(\mathbf{x}_s)| = |\dot{x}_i(s)| \leq \sqrt{\varepsilon}\ ,\ s \in [t-2q\,,t]. \tag{15}$$

Taking the derivative of $\dot{x}_i(t)$ with respect to t,

$$\ddot{x}_i(t) = \frac{\partial \dot{x}_i(t)}{\partial t} = \dot{\theta}_i(\mathbf{x}_t) = \frac{d^2 U_i(x_i)}{dx_i^2}\dot{x}_i(t) - \sum_{l \in R_i} \frac{\partial f_l(\bar{x}_l^i(t-r))}{\partial \bar{x}_l^i} \sum_{j \in \mathscr{M}_l} \dot{x}_j(t - r_i - r_j). \tag{16}$$

Let $\mathbf{d}_i := -\min_{x_i \in X} \dfrac{d^2 U_i(x_i)}{dx_i^2} > 0$. Using (15) and (16), it is possible to bound $\dot{\theta}_i(\mathbf{x}_s)$ and $\ddot{x}_i(s)$ on $s \in [t-q,t]$ with

$$|\dot{\theta}_i(\mathbf{x}_s)| = |\ddot{x}_i(s)| \leq \mathbf{d}_i |\dot{x}_i(s)| + \sum_{l \in R_i} \frac{\partial f_l(\bar{x}_l^i(s-r))}{\partial \bar{x}_l^i} |\bar{x}_l^i(s-r)| \leq (\mathbf{d}_i + \sum_{l \in R_i} M_l \alpha_l)\sqrt{\varepsilon}. \tag{17}$$

To simplify the notation, define

$$y_i := \mathbf{d}_i + \sum_{l \in R_i} M_l \alpha_l.$$

Hence, the following bound on $\theta_i(\mathbf{x}_s),\ s \in [t-q,t]$ is obtained:

$$\theta_i(\mathbf{x}_t) - q y_i \sqrt{\varepsilon} \leq \theta_i(\mathbf{x}_s) \leq \theta_i(\mathbf{x}_t) + q y_i \sqrt{\varepsilon}. \tag{18}$$

Subsequently, it is shown that $V(\mathbf{x}_t(\phi))$ is non-increasing, and a contradiction is obtained to the initial hypothesis that the set Φ is not empty. Assume that $\partial f_l(\bar{x}_l^i(t-r))/\partial \bar{x}_l^i = \partial f_l(\bar{x}_l^j(t-r))/\partial \bar{x}_l^j$, $\forall i,j \in \mathscr{M}_l$, $\forall t$ for each link l. This assumption holds for example when f_l is linear in its argument. Let B be defined in such a way that $B^T B := A^T K A$, where the positive diagonal matrix K is defined in Section 3.2. Also define the positive diagonal matrix

$$D(\mathbf{x}) := \mathrm{diag}\left[|\mathrm{D_1(x_1)}|\,, |\mathrm{D_2(x_2)}|\,, \ldots, |\mathrm{D_M(x_M)}|\right],$$

where $D_i(\mathbf{x}) := d^2 U_i(x_i)/dx_i^2$. Then, using (18),

$$\begin{aligned} \dot{V}(\mathbf{x}_t) &= -\sum_{i=1}^{M} D_i(x_i)\theta_i^2(\mathbf{x}_t) - \sum_{i=1}^{M} \theta_i(\mathbf{x}_t) \cdot \sum_{l \in R_i} \frac{\partial f_l(\bar{x}_l^i(t-r))}{\partial \bar{x}_l^i} \sum_{j \in \mathscr{M}_l} \theta_j(\mathbf{x}_{t - r_{li} - r_{lj}}) \\ &\leq -\Theta^T D \Theta - \Theta^T B^T B \Theta + q\sqrt{\varepsilon}|\Theta^T B^T B \mathbf{y}|, \end{aligned} \tag{19}$$

where everything is evaluated at t. Now, for any fixed trajectory generated by (12), and for a frozen time t, a sufficient condition for $\dot{V}(\mathbf{x}_t) \leq 0$ is

$$q\sqrt{\varepsilon} \leq \frac{||B\Theta||^2 + ||\sqrt{D}\Theta||^2}{||B\Theta||\ ||B\mathbf{y}||},$$

where $||\cdot||$ is the Euclidean norm.

Let $k := \frac{||B\Theta||}{||B\mathbf{y}||} > 0$. Rewriting the sufficient condition one obtains

$$q\sqrt{\varepsilon} \leq k + \frac{1}{k}\mu,$$

where $\mu := \frac{||\sqrt{D}\Theta||^2}{||B\mathbf{y}||^2} > 0$. The following worst-case bound on q can be derived by a simple minimization:

$$q\sqrt{\varepsilon} \leq 2\sqrt{\mu}. \tag{20}$$

Next a lower bound on μ is derived. From (14), it follows that $||\sqrt{D}\Theta(\mathbf{x}_t)||^2 \geq \bar{d}\varepsilon$, where $\bar{d} := \min_i \min_{x_i \in X} \left|\frac{d^2U_i(x_i)}{dx_i^2}\right|$, and $\sqrt{D}$ is the unique positive definite matrix whose square is D. Furthermore,

$$||B\mathbf{y}||^2 \leq \sum_{i=1}^{M} y_i \sum_{l \in R_i} \alpha_l \sum_{j \in \mathscr{M}_l} y_j.$$

Define also the following upper-bound on y_i:

$$b := \max_i \left(\mathbf{d}_i + \sum_{l \in R_i} M_l \alpha_l \right).$$

Since $\mathbf{d}_i > 0$, one obtains $||B\mathbf{y}||^2 \leq Mb^3$, and hence

$$\mu \geq \frac{\bar{d}\varepsilon}{Mb^3}.$$

Thus, from (20) a sufficient condition for $V(\mathbf{x}_t)$ to be non-increasing is

$$q \leq \frac{2\sqrt{\bar{d}}}{\sqrt{M}b^{3/2}}, \tag{21}$$

which now holds for all $t \geq 0$. □

Based on Lemma 1, $\overline{V}(t;\phi)$ is non-increasing, $\dot{\overline{V}}(t;\phi) \leq 0$. Then, using Definition 3.1 and Theorem 3.1 of [17] global asymptotic stability of the system (12) is established. Let $S := \{\phi \in C : \dot{\overline{V}}(t;\phi) = \dot{V}(\mathbf{x}_t(\phi)) = 0\}$. From (12) and (19) it follows that

$S' = \{\phi \in C : \phi(\tau) = \mathbf{x}^*,\ -q \leq \tau \leq 0\} \subset S$, as
$\Theta(\mathbf{x}_\tau) = \dot{\mathbf{x}}(\tau) = 0 \Leftrightarrow \mathbf{x}_\tau = \mathbf{x}^* \Rightarrow \dot{V}(\mathbf{x}_\tau) = 0.$

Hence, S' is the largest invariant set in S, and for any trajectory of the system that belongs identically to S, we have $\mathbf{x}_\tau = \mathbf{x}^*$. In other words, the only solution that can stay identically in S is the unique equilibrium of the system. This, then leads to the following theorem:

Theorem 3. *Assume that*

$$\frac{\partial f_l(\bar{x}_l^i(s-r))}{\partial \bar{x}_l^i} = \frac{\partial f_l(\bar{x}_l^j(s-r))}{\partial \bar{x}_l^j}, \; \forall i, j \in \mathscr{M}_l \; \forall t.$$

Then, the unique Nash equilibrium of the network game is globally asymptotically stable on the compact set of feasible flow rate vectors, $X := \{\mathbf{x} \in \mathbb{R}^M : \mathbf{A}\mathbf{x} \leq \mathbf{C}\,, \mathbf{x} \geq 0\}$ *under the gradient algorithm*

$$\dot{x}_i(t) = \frac{dU_i(x_i(t))}{dx_i} - \sum_{l \in R_i} f_l\left(\sum_{j \in \mathscr{M}_l} x_j(t - r_{li} - r_{lj}) \right),$$

in the presence of fixed heterogeneous delays, $r_{li} \geq 0$*, for all users* $i = 1, \ldots, M$*, and links* $l \in \mathscr{L}$*, if the following condition is satisfied*

$$q \leq \frac{2\sqrt{\bar{d}}}{\sqrt{M} b^{3/2}},$$

where

$$b := \max_i \left(-\min_{x_i \in X} \frac{d^2 U_i(x_i)}{dx_i^2} + \sum_{l \in R_i} M_l \alpha_l \right),$$

and

$$\bar{d} := \min_i \min_{x_i \in X} \left| \frac{d^2 U_i(x_i)}{dx_i^2} \right|.$$

If the user reaction function is scaled by a user-independent gain constant, λ, then the i^{th} user's response is given by

$$\dot{x}_i = -\lambda \frac{\partial J_i(\mathbf{x}(t))}{\partial x_i},$$

and the sufficient condition for global stability turns out to be

$$q \leq \frac{2\sqrt{\bar{d}}}{\sqrt{M} \lambda^{3/2} b^{3/2}}.$$

Notice that, for any $\lambda < 1$, the upper-bound on maximum RTT, q, is relaxed proportionally with $\lambda^{3/2}$.

The upper-bound on communication delays given in the sufficient condition of Theorem 3 is inversely proportional to the square root of the number of users multiplied by the cube of a gain constant. This structure is actually similar to those of local stability results reported in other studies [19, 27, 32]. The analysis above indicates a fundamental tradeoff between the responsiveness of the users gradient rate control algorithm and the stability properties of the system under communication delays.

4 Primal-Dual Rate Control

The distributed structure of the Internet makes it difficult, if not impossible, for users to obtain detailed real time information on the state of the network. Therefore, users are bound to use indirect aggregate metrics that are available to them, such as packet drop rate or variations in the average round trip time (RTT) of packets in order to infer the current situation in the network. Packet drops, for example, are currently used by most widely deployed versions of TCP as an indication of congestion. An approach similar to the one discussed in this section has been suggested in a version of TCP, known as TCP Vegas [13]. Although TCP Vegas is more efficient than a widely used version of TCP, TCP Reno [28], the suggested improvements are empirical and based on experimental studies.

This section presents and analyzes a primal-dual rate control scheme based on variations in the RTT a user experiences based on [5]. Although users are associated with cost functions in a way similar to the game in Section 3, the formulation here is not a proper game as the users ignore their own effects on the outcome when making decisions. Consequently, the solution here is different from the concept of Nash equilibrium.

The equilibrium solution discussed in this section maximizes the sum of user utilities under capacity constraints. The result immediately follows from a Lagrangian analysis and the concept of shadow prices. Furthermore, the solution becomes proportionally fair under logarithmic user utilities. A detailed analysis can be found in [20, 21]

4.1 Extended Network Model

An important indication of congestion for internet-style networks is the variation in queueing delay, d, which is defined as the difference between the actual delay experienced by a packet, d^a, and the fixed propagation delay of the connection, d^p. If the incoming flow rate to a router l exceeds its capacity, packets are queued (generally on a first-come first-serve basis). in the existing buffer of the router of the link with $b_{l,max}$ being the maximum buffer size. Furthermore, if the buffer of the link is full, incoming packets have to be dropped. Let the total flow on link l be given by $\bar{x}_l := \sum_{i:l\in R_i} x_i$. Thus, the buffer level at link l evolves in accordance with

$$\frac{\partial b_l(t)}{\partial t} = \begin{cases} [\bar{x}_l - C_l]^-, & \text{if } b_l(t) = b_{l,max} \\ \bar{x}_l - C_l, & \text{if } 0 < b_l(t) < b_{l,max} \\ [\bar{x}_l - C_l]^+, & \text{if } b_l(t) = 0 \end{cases} \tag{22}$$

where $[.]^+$ represents the function $\max(.,0)$ and $[.]^-$ represents the function $\min(.,0)$.

An increase in the buffers leads to an increase in the RTT of packets. Hence, RTT on a congested path is larger than the base RTT, which is defined as the sum of propagation and processing delays on the path of a packet. The queueing delay at the l^{th} link, d_l, is a nonlinear function of the excess flow on that link, given by

$$\dot{d}_l(\mathbf{x},t) = \begin{cases} \left[\frac{1}{C_l}(\bar{x}_l - C_l)\right]^-, & \text{if } d_l(t) = d_{l,max} \\ \frac{1}{C_l}(\bar{x}_l - C_l), & \text{if } 0 < d_l(t) < d_{l,max} \\ \left[\frac{1}{C_l}(\bar{x}_l - C_l)\right]^+, & \text{if } d_l(t) = 0 \end{cases} \quad (23)$$

in accordance with the buffer model described in (22), with $d_{l,max} := b_{l,max}/C_l$ being the maximum possible queueing delay. Here, $\dot{d}_l$ denotes $(\partial d_l(t)/\partial t)$. Thus, the total queueing delay, D_i, a user experiences is the sum of queueing delays on its path, namely $D_i(\mathbf{x},t) = \sum_{l \in R_i} d_l(\mathbf{x},t)$, $i \in \mathscr{M}$, which we henceforth write as $D_i(t)$, $i \in \mathscr{M}$.

4.1.1 Assumptions

Additional assumptions of the extended model presented are:

1. The effect of individual packet losses on the flow rates are ignored. This approximation is reasonable as one of the main goals of the developed rate control scheme is to minimize or totally eliminate packet losses.
2. The utility function $U_i(x_i)$ of the i^{th} user is assumed to be strictly increasing and concave in x_i.
3. The effect of a user i on the delay, $D_i(t)$, s/he experiences is ignored. This assumption can be justified for networks with a large number of users, where the effect of each user is vanishingly small. Furthermore, from a practical point of view, it is extremely difficult, if not impossible, for a user to estimate its own effect on queueing delay.

4.2 Equilibrium Solution

As in Section 3, define a cost function for each user as the difference between pricing and utility functions. However, here the pricing function of the i^{th} user is linear in x_i for each fixed total queueing delay D_i of the user, and is linear in D_i with a fixed x_i, i.e. it is a bi-linear function of x_i and D_i. The utility function $U_i(x_i)$ is assumed to be strictly increasing, differentiable, and strictly concave in a similar way and it basically describes the user's demand for bandwidth. Accordingly, variations in RTT are utilized as the basis for the rate control algorithm. The cost (objective) function for the i^{th} user at time t is thus given by

$$J_i(\mathbf{x},t) = \alpha_i D_i(t)\, x_i - U_i(x_i)\ , \quad (24)$$

which s/he wishes to minimize. In accordance with this objective, again a gradient-based dynamic model is considered where each user changes its flow rate in proportion with the gradient of its cost function with respect to its flow rate, $\dot{x}_i = -\partial J_i(\mathbf{x})/\partial x_i$.

Taking into consideration also the boundary effects, the rate control algorithm for the i^{th} user is:

$$\dot{x}_i = \begin{cases} \left[\dfrac{dU_i(x_i)}{dx_i} - \alpha_i D_i(t)\right]^- , & \text{if } x_i = x_{i,max} \\ \dfrac{dU_i(x_i)}{dx_i} - \alpha_i D_i(t), & \text{if } 0 < x_i < x_{i,max} \\ \left[\dfrac{dU_i(x_i)}{dx_i} - \alpha_i D_i(t)\right]^+ , & \text{if } x_i = 0. \end{cases} \tag{25}$$

Then, for a general network topology with multiple links, the generalized system is described by

$$\begin{aligned} \dot{x}_i(t) &= \frac{dU_i(x_i)}{dx_i} - \alpha_i D_i(t) \ , \ i = 1, \ldots, M \ , \\ \dot{d}_l(t) &= \frac{\bar{x}_l}{C_l} - 1 \ , \ l = 1, \ldots, L \ , \end{aligned} \tag{26}$$

with the boundary behavior given by (23) and (25). Define the feasible set Ω (as before) as

$$\Omega = \{(\mathbf{x}, \mathbf{d}) \in \mathbb{R}^{M+L} : 0 \le x_i \le x_{i,max} \text{and } 0 \le d_l \le d_{l,max} \ , \ \forall i \ , l\},$$

where $d_{l,max}$ and $x_{i,max}$ are upper bounds on d_l and x_i, respectively. Define $\mathbf{d}_{max} := [d_{1,max}, \ldots, d_{L,max}]$.

Existence and uniqueness of an inner equilibrium on the set Ω is now investigated under the assumption of $x_{i,max} > C_l, \forall l$. Toward this end, assume that $\mathbf{A}$ is a full row rank matrix with $M \ge L$, without any loss of generality. This is motivated by the fact that non-bottleneck links on the network have no effect on the equilibrium point, and can safely be left out.

Theorem 4. *Let $0 \le \alpha_{i,min} \le \alpha_i \le \alpha_{i,max}$, $\forall i \in \mathscr{M}$ where the elements of the vector α_{max} are arbitrarily large, and $\mathbf{A}$ be of full row rank. Given X, if α_{min} and $\mathbf{d}_{max}$ satisfy*

$$0 < \max_{\mathbf{x} \in X} \mathbf{d}(\alpha_{min}, \mathbf{x}) < \mathbf{d}_{max},$$

where $\mathbf{d}(\alpha, \mathbf{x})$ is defined in (30), then the system (26) has a unique equilibrium point, $(\mathbf{x}^*, \mathbf{d}^*)$, which is in the interior of the set Ω.

Proof. Supposing that (26) admits an inner equilibrium and by setting $\dot{x}_i(t)$ and $\dot{d}_l(t)$ equal to zero for all l and i one obtains

$$\mathbf{A}\,\mathbf{x} = \mathbf{C} \tag{27}$$

$$\mathbf{f}(\alpha, \mathbf{x}) = \mathbf{A}^T \mathbf{d} \ , \tag{28}$$

where $\mathbf{d} := [d_1, \ldots, d_L]^T$ is the delay vector at the links, $\mathbf{C}$ is the capacity vector introduced earlier, and the nonlinear vector function $\mathbf{f}$ is defined as

$$\mathbf{f}(\alpha, \mathbf{x}) := \left[\frac{1}{\alpha_1} \frac{dU_1}{dx_1}, \ldots, \frac{1}{\alpha_M} \frac{dU_M}{dx_M} \right]^T . \tag{29}$$

Define $X := \{\mathbf{x} \in \mathbb{R}^M : \mathbf{A}\mathbf{x} = \mathbf{C}\}$ as the set of flows, $\mathbf{x}$, which satisfy (27).

Multiplying (28) from left by $\mathbf{A}$ yields

$$\mathbf{A}\,\mathbf{f}(\alpha, \mathbf{x}^*) = \mathbf{A}\mathbf{A}^T \mathbf{d}.$$

Since $\mathbf{A}$ is of full row rank, the square matrix $\mathbf{A}\mathbf{A}^T$ is full rank, and hence invertible. Thus, for a given flow vector $\mathbf{x}$ and pricing vector α,

$$\mathbf{d}(\alpha, \mathbf{x}) = (\mathbf{A}\mathbf{A}^T)^{-1} \mathbf{A}\mathbf{f}(\alpha, \mathbf{x}), \tag{30}$$

is unique. From the definition of $\mathbf{f}$, $\mathbf{d}(\alpha, \mathbf{x})$ is a linear combination of $p_i(x_i)/\alpha_i$, and hence, strictly decreasing in α. Since the set X is compact, the continuous function $\mathbf{d}(\alpha, \mathbf{x})$ admits a maximum value on the set X for a given α. Therefore, for each $\varepsilon > 0$ one can choose the elements of α_{max} sufficiently large such that

$$0 < \max_{\mathbf{x} \in X} \mathbf{d}(\alpha_{max}, \mathbf{x}) < \varepsilon.$$

In addition, given X and $\mathbf{d}_{max}$, one can find α_{min} such that

$$0 < \max_{\mathbf{x} \in X} \mathbf{d}(\alpha_{min}, \mathbf{x}) < \mathbf{d}_{max}, \tag{31}$$

Hence, there is at least one inner equilibrium solution, $(\mathbf{x}^*, \mathbf{d}^*)$, on the set Ω, which satisfies (27) and (28).

The uniqueness of the equilibrium is established next. Suppose that there are two different equilibrium points, $(\mathbf{x}_1^*, \mathbf{d}_1^*)$ and $(\mathbf{x}_2^*, \mathbf{d}_2^*)$. Then, from (27) it follows that

$$\mathbf{A}\,(\mathbf{x}_1^* - \mathbf{x}_2^*) = 0 \;\Leftrightarrow\; (\mathbf{x}_1^* - \mathbf{x}_2^*)^T \mathbf{A}^T = 0.$$

Similarly, from (28) follows

$$\mathbf{f}(\alpha, \mathbf{x}_1^*) - \mathbf{f}(\alpha, \mathbf{x}_2^*) = \mathbf{A}^T (\mathbf{d}_1^* - \mathbf{d}_2^*) \,.$$

Multiplying this with $(\mathbf{x}_1^* - \mathbf{x}_2^*)^T$ from left one obtains

$$(\mathbf{x}_1^* - \mathbf{x}_2^*)^T \left[\mathbf{f}(\alpha, \mathbf{x}_1^*) - \mathbf{f}(\alpha, \mathbf{x}_2^*)\right] = 0$$

Rewrite this as

$$\sum_{i=1}^{M} (\mathbf{x}_{1i}^* - \mathbf{x}_{2i}^*) \frac{1}{\alpha_i} \left[\frac{dU_i(x_{1i}^*)}{dx_i} - \frac{dU_i(x_{2i}^*)}{dx_i} \right] = 0.$$

Since U_i's are strictly concave, each term (say the i-th one) in the summation is negative whenever $x^*_{1i} \neq x^*_{2i}$ with equality holding only if $x^*_{1i} = x^*_{2i}$. Hence, the point x^* has to be unique, that is $\mathbf{x}^* = \mathbf{x}^*_1 = \mathbf{x}^*_2$. From this, and (26), it immediately follows that $D_i, i = 1, \ldots, M$, are unique. This does not however immediately imply that $d_l, l = 1, \ldots, L$, are also unique, which in fact may not be the case if $\mathbf{A}$ is not full row rank. The uniqueness of d_l's, however, follow from (30), where a unique $\mathbf{d}^*$ is obtained for a given equilibrium flow vector $\mathbf{x}^*$:

$$\mathbf{d}^* = (\mathbf{A}\mathbf{A}^T)^{-1}\mathbf{A}\mathbf{f}(\alpha, \mathbf{x}^*).$$

Thus, $(\mathbf{x}^*, \mathbf{d}^*)$, following from (27) and (28), constitutes a unique inner equilibrium point on the set Ω. □

4.3 Stability Analysis

The rate control scheme and accompanying system described by (26) is first shown to be globally asymptotically stable under a general network topology in the ideal case. Subsequently, the global stability of the system is investigated under arbitrary *information delays*, denoted by r, for a general network with a single bottleneck node and multiple users. The case of multiple users on a general network topology with multiple links is omitted since the problem in that case is quite intractable under arbitrary information delays.

4.3.1 Instantaneous Information Case

The stability of the system below can easily be established under the assumption that users have instantaneous information about the network state. Alternatively, this case can be motivated by assuming that information delays are negligible in terms of their effects to the rate control algorithm.

Defining the delays at links, d_l, and user flow rates, x_i, around the equilibrium as $\tilde{d}_l := d_l - d^*_l$ and $\tilde{x}_i := x_i - x^*_i$, respectively, for all l and i, one obtains the following system inside the set Ω and around the equilibrium:

$$\begin{aligned} \dot{\tilde{x}}_i(t) &= g_i(\tilde{x}_i) - \alpha_i \tilde{D}_i(t) \ , \ i = 1, \ldots, M \ , \\ \dot{\tilde{d}}_l(t) &= \frac{1}{C_l} \sum_{i:l \in R_i} \tilde{x}_i \ , \ l = 1, \ldots, L \ , \end{aligned} \tag{32}$$

where $\tilde{D}_i = \sum_{l \in R_i} \tilde{d}_l$, and $g_i(\tilde{x}_i)$ is defined as

$$g_i(\tilde{x}_i) := \frac{dU_i(x_i)}{dx_i} - \frac{dU_i(x^*_i)}{dx_i}.$$

Define next a positive definite Lyapunov function

$$V(\tilde{\mathbf{x}}, \tilde{\mathbf{d}}) = \sum_{i=1}^{M} \frac{1}{\alpha_i} (\tilde{x}_i)^2 + \sum_{l=1}^{L} C_l (\tilde{d}_l)^2. \qquad (33)$$

The time derivative of $V(\tilde{\mathbf{x}}, \tilde{\mathbf{d}})$ along the system trajectories is given by

$$\dot{V}(\tilde{\mathbf{x}}, \tilde{\mathbf{d}}) = \sum_{i=1}^{M} (2/\alpha_i) g_i(\tilde{x}_i) \tilde{x}_i \leq 0,$$

where the inequality follows because $g_i(\tilde{x}_i)\tilde{x}_i \leq 0$, $\forall i$. Thus, $\dot{V}(\tilde{\mathbf{x}}, \tilde{\mathbf{d}})$ is negative semidefinite. Let $S := \{(\tilde{\mathbf{x}}, \tilde{\mathbf{d}}) \in \mathbb{R}^{M+L} : \dot{V}(\tilde{\mathbf{x}}, \tilde{\mathbf{d}}) = 0\}$. It follows as before that $S = \{(\tilde{\mathbf{x}}, \tilde{\mathbf{d}}) \in \mathbb{R}^{M+L} : \tilde{\mathbf{x}} = 0\}$. Hence, for any trajectory of the system that belongs identically to the set S, we have $\tilde{\mathbf{x}} = 0$. It follows directly from (32) and the fact that $g_i(0) = 0 \; \forall i$ that

$$\tilde{\mathbf{x}} = 0 \Rightarrow \dot{\tilde{\mathbf{x}}} = 0 \Rightarrow \tilde{D}_i = 0 \; \forall i \Rightarrow \tilde{d}_l = 0 \; \forall l,$$

where the last implication is due to the fact that $\tilde{D} = \mathbf{A}^T \tilde{\mathbf{d}}^*$ and the matrix $\mathbf{A}$ is of full row rank. Therefore, the only solution that can stay identically in S is the zero solution, which corresponds to the unique inner equilibrium of the original system. As a result, the system (32) is globally stable under the assumption of instantaneous information.

4.3.2 Information Delay Case

The preceding analysis is generalized to account for information delays in the system by introducing user specific maximum propagation delays $r = [r_1, \ldots, r_M]$ between a bottleneck link and the users. The system is assumed to have a unique inner equilibrium point $(\mathbf{x}^*, d^*)$ as characterized in Section 4.2. Modifying the system equations around this equilibrium point by introducing the associated maximum propagation delays, one obtains

$$\begin{aligned} \dot{\tilde{x}}_i(t) &= g_i(\tilde{x}_i(t)) - \alpha_i \tilde{d}(t - r_i) \;, \; i = 1, \ldots, M \\ \dot{\tilde{d}}(t) &= \frac{1}{C} \sum_{i=1}^{M} \tilde{x}_i(t - r_i). \end{aligned} \qquad (34)$$

Then, the i^{th} users rate control algorithm is

$$\dot{\tilde{x}}_i(t - r_i) = g_i(\tilde{x}_i(t - r_i)) - \alpha_i \tilde{d}(t) + \frac{\alpha_i}{C} \int_{-2r_i}^{0} \sum_{j=1}^{M} \tilde{x}_j(t + s - r_j) ds.$$

Define again a positive definite Lyapunov function:

$$V(\tilde{\mathbf{x}}, \tilde{d}) = \sum_{i=1}^{M} \frac{1}{\alpha_i} (\tilde{x}_i(t - r_i))^2 + C(\tilde{d}(t))^2 + \frac{M}{C} \sum_{i=1}^{M} \int_{-2r_i}^{0} \int_{t+s}^{t} \tilde{x}_i^2(u - r_i) du \, ds. \qquad (35)$$

Taking the derivative of V along the system trajectories yields

$$\begin{aligned}\dot{V}(\tilde{\mathbf{x}},\tilde{d}) &= \textstyle\sum_{i=1}^{M} \frac{2}{\alpha_i} g_i(\tilde{x}_i(t-r_i))\tilde{x}(t-r_i) \\ &+ \frac{1}{C}\textstyle\int_{-2r_i}^{0} \sum_{i=1}^{M} \sum_{j=1}^{M} 2\tilde{x}_i(t-r_i)\tilde{x}_j(t+s-r_j)ds \\ &+ \frac{M}{C}\textstyle\sum_{i=1}^{M} \int_{-2r_i}^{0} [\tilde{x}_i^2(t-r) - \tilde{x}_i^2(t+s-r)]ds.\end{aligned}$$

This derivative $\dot{V}$ is bounded from above by

$$\dot{V}(\tilde{\mathbf{x}},\tilde{d}) \le \sum_{i=1}^{M} \frac{2}{\alpha_i} g_i(\tilde{x}_i(t-r_i))\tilde{x}_i(t-r_i) + \frac{4Mr_i}{C}\tilde{x}_i^2(t-r_i).$$

Hence, it can be made negative semi-definite by imposing a condition on the maximum delay in the system, $r_{max} := \max_i r_i$. Let $S := \{(\tilde{\mathbf{x}},\tilde{d}) \in \tilde{\Omega} : \dot{V}(\tilde{\mathbf{x}},\tilde{d}) = 0\}$. It follows as before that $S = \{(\tilde{\mathbf{x}},\tilde{d}) \in \tilde{\Omega} : \tilde{\mathbf{x}} = 0\}$. Therefore, for any trajectory of the system that belongs identically to the set S, $\tilde{\mathbf{x}} = 0$. It also follows directly from (34), and the fact that $g_i(0) = 0\ \forall i$, that

$$\tilde{\mathbf{x}} = 0 \Rightarrow \dot{\tilde{\mathbf{x}}} = 0 \Rightarrow \tilde{d} = 0,$$

where the fact that the matrix $\mathbf{A}$ is of full row rank is used. Consequently, the only solution that can stay identically in S is the zero solution, which corresponds to the unique equilibrium of the original system. As a result, the system (34) is asymptotically stable by LaSalle's invariance theorem [22] if the maximum delay in the system, r_{max}, satisfies the condition

$$r_{max} < \frac{k_{min}}{2\alpha_{max}}\frac{C}{M}, \tag{36}$$

where α_{max} and k_{min} are defined as

$$\begin{aligned}\alpha_{max} &:= \max_i \alpha_i \\ k_{min} &:= \min_i \inf_{-x_i^* \le \tilde{x}_i \le x_{i,max} - x_i^*} \left|\frac{g(\tilde{x}_i)}{\tilde{x}_i}\right|.\end{aligned} \tag{37}$$

The following theorem summarizes this result:

Theorem 5. *Let the conditions in Theorem 4 hold such that the system*

$$\begin{aligned}\dot{x}_i(t) &= \frac{dU_i(x_i(t))}{dx_i} - \alpha_i d(\mathbf{x}, t-r),\ i = 1,\dots,M, \\ \dot{d}(t) &= \frac{1}{C}\sum_{i=1}^{M} x_i(t-r_i) - 1,\end{aligned}$$

admits a unique inner equilibrium point $(\mathbf{x}^*, d^*)$. This system is globally asymptotically stable, if the maximum delay, r_{max}, in the system satisfies the condition

$$r_{max} < \frac{k_{min}}{2\alpha_{max}} \frac{C}{M},$$

where α_{max} *and* k_{min} *are defined in (37).*

Notice that the bound on the maximum delay required for the stability of the system is affected by, among other things, the maximum pricing parameter and the capacity per user C/M. Since the link capacity C will be provisioned in the network design stage according to the expected maximum number of users the proposed algorithm is in practice scalable for the given capacity per user.

5 Robust Rate Control

The link capacities C on a network fluctuate due to short-lived background traffic as well as due to the inherent characteristics of the network, e.g. as a result of fading in the case of wireless networks. Relaxing the assumption on the knowledge of the available bandwidth B at a bottleneck link, it is possible to define a function of it, $w(B)$, simply as an input to end users instead of attempting to model it explicitly. Then, define a system from the perspective of a user $i \in \mathscr{M}$ which keeps track of the available bandwidth on a bottleneck link shared by $M-1$ others. The system state s_i reflects from the perspective of device i roughly the bandwidth availability on its path. Then, the system equation for user i is

$$\dot{s}_i = a\, s_i + b\, u_i + w, \tag{38}$$

where u_i represents the *control action* of the user. The parameters $a < 0$ and $b < 0$ adjust the memory horizon (the smaller a the longer the memory) and the "expected" effectiveness of control actions, respectively, on the system state s_i. The user i bases its control actions on its state which not only takes as input the current available bandwidth but also accumulates the past ones to some extent. It is also possible to interpret the system (38) as a low pass filter with input w and output s.

Based on the discussion above, the following rate control scheme which is approximately proportional to the control actions, is proposed:

$$\dot{x}_i = -\phi x_i + u_i, \tag{39}$$

where $\phi > 0$ is sufficiently small. Although this rate update scheme seems disconnected from the system in (38) it is not the case as we show in the next section. As a result of w being a function of the available bandwidth $B = C - \sum_{k=1}^{M} x_k$, which in turn is a function of the link capacity and aggregate user rates, the systems (38) and (39) are connected via a feedback loop.

For simplicity, the coefficient of u_i is chosen to be 1 in (39). Since a rate update of a device will have the inverse effect on the available bandwidth, the parameter b in (38) is naturally picked to be negative. Notice that this is a "bandwidth probing scheme" in a sense similar to additive-increase multiplicative-decrease (AIMD) feature of the well-known transfer control protocol (TCP). However, in this case the user decides on the rate control action by solving an optimization problem. The objectives of this problem include full bandwidth-utilization while preventing excessive rate fluctuations leading to instabilities and jitter.

5.1 Equilibrium and Stability Analysis for Fixed Capacity

In order to compute the control actions u given the state s, consider a linear feedback control scheme of the general form $u = \theta s$, where θ is a positive constant. An equilibrium and stability analysis of the system (38) and (39) is conducted under this general class of linear feedback controllers for a single bottleneck link of fixed capacity C shared by M users. The analysis of this special fixed-capacity case provides valuable insights to the original problem.

By ignoring the noise in the system, make the simplifying assumption of $w := C - \sum_{i=1}^{M} x_i$. Then,

$$\begin{aligned} \dot{s}_i &= a s_i + b\,\theta s_i + C - \textstyle\sum_{k=1}^{M} x_k \\ \dot{x}_i &= -\phi x_i + \theta s_i, \quad i = 1, \ldots, M. \end{aligned} \tag{40}$$

At the equilibrium -which is unique, and is asymptotically stable- $\dot{s}_i = \dot{x}_i = 0\ \forall i$. Solving for equilibrium values of s_i and x_i for all i, denoted by s_i^* and x_i^*, respectively, one obtains

$$x_i^* = \frac{C\theta}{\theta M - (a + b\theta)\phi}$$

and

$$s_i^* = \frac{C\phi}{\theta M - (a + b\theta)\phi},$$

which are unique, under the negativity of a and b and positivity of θ, as long as $\phi > 0$. As $\phi \to 0^+$, it follows that $\sum_i x_i \to C$. Thus, as ϕ approaches to zero from the positive side, linear feedback controllers of the form $u = \theta s$, where $\theta > 0$ ensure maximum network usage when the capacity C is fixed and there is no noise. Notice that the equilibrium rate x^* is on the order of C/M and usually much larger than zero, which constitutes a physical boundary due to nonnegativity constraint.

It is next proven that the linear system (40) is stable and asymptotically converges to the equilibrium point whenever $\phi > 0$. Toward this end, sum the rates x_i in (40) to obtain

$$\begin{aligned} \dot{s}_i &= -\mu s_i - \bar{x} + C, \quad i = 1, \ldots, M \\ \dot{\bar{x}} &= -\phi \bar{x} + \theta \textstyle\sum_{i=1}^{M} s_i, \end{aligned} \tag{41}$$

where $\bar{x} := \sum_{i=1}^{M} x_i$ and $\mu := -(a + b\theta) > 0$. We can rewrite (41) in the matrix form as

$$\dot{y} = Fy + [C \cdots C\ 0]^T,$$

where $y := [s_1 \cdots s_M \; \bar{x}]^T$. Then, it is straightforward to show that the characteristic function of the $(M+1)$ dimensional square matrix F has the form

$$\det(\lambda I - F) = (\lambda + \mu)^{M-1} \left[\lambda^2 + (\mu + \phi)\lambda + \mu\phi + M\theta\right] = 0.$$

Notice that, F has $M-1$ repeated negative eigenvalues at $\lambda = -\mu$ and two additional eigenvalues at

$$\lambda_{1,2} = -\frac{1}{2}(\mu + \phi) \pm \frac{1}{2}\sqrt{(\mu - \phi)^2 - 4M\theta}.$$

If $(\mu - \phi)^2 < 4M\theta$, then both of these eigenvalues are imaginary with negative real parts. Otherwise, we have $\mu + \phi > |\mu - \phi|$ and both eigenvalues are negative and real. Therefore, all eigenvalues of F always have negative real parts and the linear system (41) is stable.

It immediately follows that s_i is always finite and converges to the equilibrium, and from the second equation of (40), x_i has to be finite and converges for all i. Thus, the original system (40) is stable.

5.2 H^∞-Optimal Rate Control

Having obtained thereof equilibrium state of (40) and shown its asymptotic stability for fixed capacity, it is analyzed for robustness. First, rewrite the system (40) around the equilibrium point (s_i^*, x_i^*) $\forall i$ to obtain

$$\begin{aligned} \dot{\tilde{s}}_i &= a\,\tilde{s}_i + b\,\tilde{u}_i + w \\ \dot{\tilde{x}}_i &= -\phi\tilde{x}_i + \theta\tilde{s}_i, \quad i = 1, \ldots, M, \end{aligned} \tag{42}$$

where $\tilde{s}_i := s_i - s_i^*$, $\tilde{x}_i := x_i - x_i^*$, and $\tilde{u}_i := u_i - u_i^*$. Then, reformulate the rate control objectives described earlier within a disturbance rejection problem around the equilibrium. Subsequently, H$^\infty$ optimal control theory allows for removal of all the simplifying assumptions of the previous subsection on w and solve the problem in the most general case. By viewing the disturbance (here the available bandwidth) as an intelligent maximizing opponent in a dynamic zero-sum game who plays with knowledge of the minimizer's control action, the system is evaluated under the worst possible conditions (in terms of capacity usage). Then, users determine their control actions that will minimize costs or achieve the objectives defined under these worst circumstances [9], resulting in a robust linear feedback rate control scheme.

Notice that, a time scale separation is assumed to exist between the variations in capacity $C(t)$ and the rate updates $x(t)$. With a sufficiently high update frequency each device can track the variations at the equilibrium point caused by the random capacity fluctuations. The robustness properties of H$^\infty$ optimal controller also play a positive role here.

The system (38) can be classified as continuous-time with perfect state measurements due to the state $\tilde{s}_i$ being an internal variable of user i. Next, an H$^\infty$-optimal control analysis and design is provided by taking this into account. First, introduce the *controlled output*, $z_i(t)$, as a two dimensional vector:

$$z_i(t) := [h\tilde{s}_i(t) \;\; g\tilde{u}_i(t)]^T, \tag{43}$$

where g and h are positive preference parameters. The cost of a user that captures the objectives defined and for the purpose of H$^\infty$ analysis is the ratio of the L^2-norm of z_i to that of w:

$$L_i(\tilde{x}_i, \tilde{u}_i, w) = \frac{\|z_i\|}{\|w\|}, \tag{44}$$

where $\|z_i\|^2 := \int_0^\infty |z_i|^2 \, d\tau$, and a similar definition applies to $\|w\|^2$. Although being a ratio, L_i is referred to as the (user) cost in the rest of the analysis. It captures the proportional changes in z_i due to changes in w. If $\|w\|$ is very large, the user cost L_i should be low even if $\|z_i\|$ is large as well. A large $\|z_i\|$ indicates that the state $|\tilde{s}_i|$ and/or the control $|u_i|$ have high values reflecting and reacting to the situation, respectively. However, they should not grow unbounded, which is ensured by a low cost, L_i. For the rest of the analysis, the subscript i denoting the user i is dropped for ease of notation.

H$^\infty$-optimal control theory guarantees that a performance factor will be met. This factor γ, also known as the H$^\infty$ norm, can be thought of as the worst possible value for the cost L. It is bounded below by

$$\gamma^* := \inf_{\tilde{u}} \sup_{w} L(\tilde{u}, w), \tag{45}$$

which is the lowest possible value for the parameter γ. It can also be interpreted as the optimal performance level in this H$^\infty$ context.

In order to solve for the optimal controller $\mu(\tilde{s})$, a corresponding (soft-constrained) differential game is defined, which is parametrized by γ,

$$J_\gamma(\tilde{u}, w) = \|z\|^2 - \gamma^2 \|w\|^2. \tag{46}$$

The environment is assumed to maximize this cost function (as part of the worst-case analysis) while the objective of the user is to minimize it. The optimal control action $\tilde{u} = \mu_\gamma(\tilde{s})$ can be determined from this differential game formulation for any $\gamma > \gamma^*$.

This controller is expressed in terms of a relevant solution, σ_γ, of a related game algebraic Riccati equation (GARE) [9]:

$$2a\sigma - \left(\frac{b^2}{g^2} - \frac{1}{\gamma^2}\right)\sigma^2 + h^2 = 0. \tag{47}$$

By the general theory [12], the relevant solution of the GARE is the "minimal" one among its multiple nonnegative-definite solutions. However, in this case, since the GARE is scalar, and the system is open-loop stable (that is, $a < 0$), the GARE (which is a quadratic equation) admits a unique positive solution for all $\gamma > \gamma^*$, and the value of γ^* can be computed explicitly in terms of the other parameters. Solving for the roots of (7):

$$\sigma_\gamma = \frac{-a \pm \sqrt{a^2 - \lambda h^2}}{\lambda}$$

where

$$\lambda := \frac{1}{\gamma^2} - \frac{b^2}{g^2}.$$

The parameter λ could be both positive and negative, depending on the value of γ, but for γ close in value to γ^* it will be positive. Further, γ^* is the smallest value of γ for which the GARE has a real solution. Hence,

$$\gamma^* = \left[\sqrt{\frac{a^2}{h^2} + \frac{b^2}{g^2}}\right]^{-1}.$$

Finally, a controller that guarantees a given performance bound $\gamma > \gamma^*$ is:

$$u_\gamma = \mu_\gamma(s) = -\left(\frac{b}{g^2}\,\sigma_\gamma\right) s. \tag{48}$$

This is a stabilizing linear feedback controller operating on the device system state s, where the gain can be calculated offline using only the linear quadratic system model and for the given system and cost parameters.

It is important to note that although the analysis and controller design are conducted around the equilibrium point, the users do not have to compute the actual equilibrium values. In other words, (48) can be equivalently written in terms of $\tilde{u}_\gamma$ and $\tilde{x}$. In practice, the H^∞-optimal rate control scheme is implemented as follows: each user i keeps track of the measured available bandwidth B_i on the network via the state equation (38), which takes the respective $w_i(B_i)$ as input. Then, the linear feedback control u_i is computed in (48) for a given set of system (a,b) and preference (h,g) parameters. Finally, each user updates its flow rate using (39) on each network. A discretized version of the algorithm is summarized in Figure 1.

Input: Available bandwidth B measurements on the network;
Parameters: User preferences (a,b) and (h,g);
Output: Feedback control u for each user and rate x;

Measure current available bandwidth (w) ;
Update s using (38) ;
Compute u using (48) ;
Update flow rate x using (39) ;

Fig. 1 H^∞-optimal rate control scheme from a user perspective.

6 Discussion

The multi-faceted and complex nature of the rate allocation and control problem allows for a variety of approaches and diverse solutions. The objectives of efficiency, fairness, and incentive-compatibility can be formulated in a variety of ways and can even be conflicting in many cases. Furthermore, the network models used for analysis may abstract different aspects of the underlying complex networks. It is, therefore, not surprising to observe the existence of a huge literature on the topic.

Section 3 has presented a game-theoretic framework that addresses the cases where users on the network are selfish and noncooperative. In other words, users do not follow a cooperative protocol out of goodwill as in TCP and decrease their flow rates voluntarily when there is a congestion, but want as much bandwidth as they can get in all situations. To prevent undesirable outcomes such as congestion collapse, pricing is proposed as an enforcement mechanism. The solution concept adopted is the Nash equilibrium, where no user has an incentive to deviate from it.

Section 4 has studied a primal-dual algorithm to solve a distributed optimization (maximization) of a global objective function, which is defined as the sum of user utilities, under capacity constraints. Its solution can be interpreted as a social optimum where the resulting flow rates are also proportionally fair for logarithmic user utilities. Here, the users are assumed to be cooperative in the sense that they follow a primal-dual distributed algorithm and ignore their own effect on the outcome when making decisions. Although this approach is mathematically similar to the one in Section 4, the solution point is not a proper Nash equilibrium, and hence the users have to be cooperative to achieve it.

Unlike the previous two sections, Section 5 has focused on robustness with respect to parameter changes in the problem instead of optimization. A well known method, H^∞-optimal control, from control theory is used to design a distributed rate control scheme, where each user measures the current system state and acts independently from others. One of the main aspects of the resulting algorithm is its adaptive nature to the variations in total available bandwidth. Such variations in capacity are more the rule rather than exception in wired networks due to short lived or unresponsive flows and in wireless networks due to channel fading (fluctuation) effects.

While the formulations in Sections 3, 4, and 5 share some common aspects, such as the underlying network model, they differ from each other in terms of their emphasis. The rate control game of Section 3 focuses mainly on incentive compatibility and adopts Nash equilibrium as the preferred solution concept. The primal-dual scheme of Section 4 solves a global optimization problem defined as the maximization of sum of user utilities. It also extends the basic fluid network model by taking into account the queue dynamics and is built upon available information to users for decision making. The robust rate control scheme of Section 5 emphasizes robustness with respect to capacity changes and delays. It also differs from the previous two, which share a utility-based approach, by focusing mainly on fully efficient usage of the network capacity rather than user utilities.

In conclusion, the three control and game-theoretic formulations presented provide diverse insights to the problem through rigorous mathematical analysis instead of focusing on a single specific system with a certain set of preferences and assumptions. Although implementation and architectural aspects have not been discussed here, it is hoped that the sound mathematical principles derived will be useful as a basis for engineering future rate control schemes.

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