Nash Equilibrium Design and Optimization

Tansu Alpcan and Lacra Pavel

***Abstract*— The general problem of Nash equilibrium design is investigated from an optimization perspective. Within this context, a specific but fairly broad class of noncooperative games are considered that have been applied to a variety of settings including network congestion control, wireless uplink power control, and optical power control. The Nash equilibrium design problem is analyzed under various knowledge assumptions (full versus limited information) and design objectives (QoS versus utility maximization). Among other results, the "price of anarchy" is shown not to be an inherent feature of games that incorporate pricing mechanisms, but merely a misconception that often stems from arbitrary choice of game parameters. Moreover, a simple linear pricing is sufficient for design of Nash equilibrium according to a chosen global objective for a general class of games and under suitable information assumptions.**

I. Introduction

Game theory has been recently enjoying immense popularity in the research community as it provides a new perspective to optimization, networking, and distributed control problems. It incorporates paradigms such as Nash equilibrium and incentive compatibility, can help quantifying individual preferences of decision-makers, and has an inherently distributed nature. Consequently, game theoretic models have been applied to a variety of problem domains ranging from economics to communication networks and security [1]–[5].

Despite a general agreement on the usefulness of game theory, there seems to be an ongoing and widespread misconception in the research community about an unavoidable "price of anarchy" or "efficiency loss" associated with any noncooperative game formulation even under the existence of pricing mechanisms. Unsurprisingly, this loss of efficiency has been the subject of many investigations [6]–[8] and a variety of pricing schemes have been proposed in the literature aiming to improve Nash equilibrium (NE) efficiency with respect to a chosen criterion in specific settings [9]–[13]. In addition, a separate but substantial literature exists under the umbrella of implementation theory, especially in the field of economics, which focuses on finding fundamental bounds for games where the outcome satisfies some given criteria [14]. The research community has revisited the issue of mechanism design only very recently and indirectly addressed some of the earlier misconceptions [15], [16]. On the other hand, these studies are limited either to special problem formulations or adopt specific efficiency criteria such as sum of user utility maximization [17].

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T. Alpcan is with Deutsche Telekom Laboratories, Berlin 10587, Germany tansu.alpcan@telekom.de

L. Pavel is with the Department of Electrical and Computer Engineering, University of Toronto, Toronto, ON, M5S 3G4, Canada pavel@control.utoronto.ca

Although tragedy of commons or price of anarchy are unavoidable in "pure" games without any external factors, they can be circumvented altogether when additional mechanisms such as "pricing" are included in the game formulation. In parallel to some earlier results [18], this paper shows that simple linear pricing is sufficient for design of NE according to a chosen global objective for a broad class of games. Therefore, *"loss of efficiency" is not an inherent feature of a broad class of games with built-in pricing systems, but merely a misconception that often stems from arbitrary choice of game parameters*.

While it is straightforward to optimize NE according to some criterion under full information, the problem is much more complicated under information and communication constraints. The game or system designer (Figure 1) usually does not have full information about the system parameters such as user preferences or utility functions. Under this kind of information constraints, the designer either deploys additional dynamic feedback mechanisms or requires side information from the system depending on the specific design objectives. An example for the former case is a dynamic pricing scheme operating as an "outer feedback loop". If the objective is to achieve a social optimum (e.g. maximization of sum of user utilities) or satisfying some quality of service (QoS) conditions, then the designer often needs limited but accurate (honest) information from users or the system. It is important to note that, if the users have the capability of manipulating such side information, then the design problem can be more involved even ill-defined. For example, the goal of reaching a social optimum without knowing true user utilities but having only access to manipulated data may not a realistic or even feasible one [14]. Although mechanisms such as VSG have been proposed to circumvent these issues, the resulting schemes are often limited and demanding in terms of communication requirements [15].

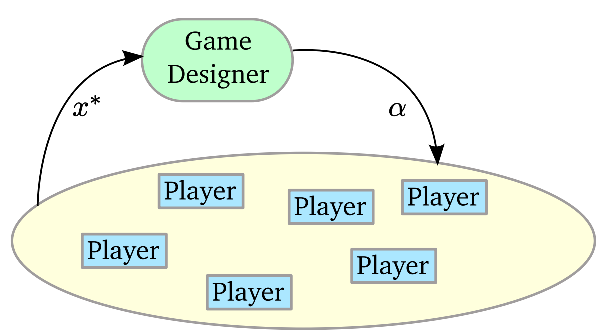

Fig. 1. The rules or pricing mechanisms within a game can be set by a "designer" to influence the outcome.

This paper -to the best of our knowledge- constitutes the only effort aiming to investigate the general problem of NE design in a broad and constructive manner *from an optimization and control theoretic perspective*. We study a fairly general class of games that have been applied in a variety of settings including network congestion, wireless uplink power, and optical power control problems. Furthermore, we investigate NE design for various knowledge assumptions (full versus limited information) and design objectives (QoS versus utility maximization). Conditions for pricing functions that allow locating the NE solutions to any desired point are derived. In addition, convergence of two example dynamic pricing schemes is shown under the time-scale separation assumption between the game and pricing dynamics. On the other hand, we restrict our treatment to a class of games where players do not manipulate the game by deceiving the system designer and where utility functions accurately reflect user preferences.

The rest of the paper is organized as follows. Section II presents the game model and NE design problem formulations. Section III discusses NE design under complete information, whereas Section IV investigates the incomplete information case with two specific objective functions: QoS-based and utility maximization. Subsequently, a brief overview of NE dynamic control is given in Section V which is followed by the concluding remarks of Section VI.

II. Model and Problem Formulation

Consider a class of N player static *noncooperative* games, denoted by $\mathcal{G}\mathbf{0}$, on the compact action (strategy) space $\Omega \subset \mathbb{R}^N$ where the i^{th} player's actions are denoted by the vector $\mathbf{x}_i$, $\mathbf{x} \in \Omega$. Furthermore, the i^{th} player is associated with a smooth (continuously differentiable) cost function, $J_i : \mathbb{R} \times \Omega \to \mathbb{R}$, $J_i(\alpha_i, \mathbf{x}),\ i = 1, 2, \ldots, N$, parametrized by a scalar "pricing" or game parameter $\alpha_i \in \mathbb{R}$. In some formulations, there may be (coupled) restrictions on the domain of these parameters such that $\alpha \in \hat{\Omega} \subset \mathbb{R}^N$. However, it will be assumed in this paper that $\hat{\Omega} = \mathbb{R}^N$ for simplicity. Assuming a set of sufficient conditions for the existence of at least one Nash equilibrium (NE) are satisfied, define a game mapping, $\mathcal{T}$ (an inverse game mapping $\hat{\mathcal{T}}$) that maps game parameters α to NE points (NE points to parameters):

$$\mathcal{T} : \mathbb{R}^N \to \Omega \quad \text{and} \quad \hat{\mathcal{T}} : \Omega \to \mathbb{R}^N, \tag{1}$$

such that

$$\mathbf{x}^* = \mathcal{T}(\alpha^*) \quad \text{and} \quad \alpha^* = \hat{\mathcal{T}}(\mathbf{x}^*) \tag{2}$$

for any NE point $\mathbf{x}^*$ and corresponding parameter vector α^*. Notice that the mappings $\mathcal{T}$ and $\hat{\mathcal{T}}$ are highly nonlinear, often not explicitly expressible, and may not be one-to-one or invertible except for special cases, i.e. games with special properties.

Next, consider a class of games, $\mathcal{G}\mathbf{1}$, by assuming a specific cost structure of the form

$$J_i(\alpha_i, \mathbf{x}) = \alpha_i p_i(\mathbf{x}) - U_i(\mathbf{x}), \tag{3}$$

where the functions p_i and U_i are smooth and chosen in such a way that there exists at least a single NE in the game, e.g. the function p_i can be convex while U_i is strictly concave with respect to x_i for any given $\mathbf{x}_{-i}$. Further define another class of games, $\mathcal{G}\mathbf{2}$, as a special case of $\mathcal{G}1$ with additional conditions on the cost structure, such that they admit a unique NE solution. An extensive analysis on conditions for the existence and uniqueness of NE can be found in [19]. Notice that a large set of network games belong to this class with notable examples of network congestion games [11], [20], power control games in wireless networks [1] and optical networks [4].

Assume that the utility function U_i accurately reflects the user preferences. Then, the pricing function p_i and parameters α enable the system designer to influence (optimize, control) the game outcome to achieve a desired objective. This is similar -in spirit- to the goal of implementation theory or mechanism design in the economics literature [14] with the important difference of not allowing users to knowingly manipulate the system. Then, the problem of designing the NE of the game can be formulated as follows, which is also illustrated in Figure 2.

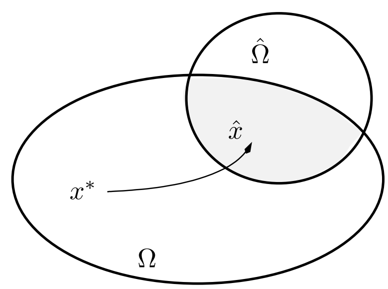

Fig. 2. The objective of NE, $\mathbf{x}^* \in \Omega$, design may involve moving it to a desirable region $\hat{\Omega} \cap \Omega$ or a specific point $\hat{\mathbf{x}}$.

Problem 1. *How to choose the pricing function p and parameters α such that the NE of games of class $\mathcal{G}1$ satisfies some desirable properties?*

Two specific but common examples of such properties are

1) The NE coincides with the solution of a global optimization problem, e.g. welfare maximization:
$$\mathbf{x}^* = \arg\max \sum_i U_i(\mathbf{x}) \ \text{ such that } \mathbf{x} \in \Omega.^1$$
2) The NE satisfies some system or user-dependent constraints such as capacity constraints, non-negativity, or performance bounds. For example, the favorable set $\hat{\Omega}$ can be defined as
$$\hat{\Omega} := \{\mathbf{x} \in \mathbb{R}^N : \mathbf{x}_i \geq 0\ \forall i,\ \sum_i \mathbf{x}_i \leq C,\ s_i(\mathbf{x}) \geq \bar{s}_i\ \forall i\},$$
where C is a capacity constraint, $s(\cdot)$ is a quality of service (QoS) measure such as signal-to-interference ratio (SIR), and $\bar{s}$ denotes the minimum acceptable QoS level.

[1] All summations in the paper are from $1, \ldots, N$ unless explicitly defined.

An important aspect of Problem 1 is the amount of knowledge available to the system designer in optimization of NE. If there is complete knowledge of player preferences and global system objective, then the approach to be adopted can be quite different from the one when the designer has very limited information. In cases when the game dynamics are very fast, it is appropriate to focus on static optimization of the NE point. Then, the actions of the system designer are assumed to be on a slower time-scale than the actual game dynamics between the players resulting in a hierarchical structure. However, if the game dynamics are slow or there are external disturbances, then the game can be treated as a dynamic control system that needs to be stabilized around the desired point.

III. NE Design under Complete Information

First, we investigate the question of NE design under the assumption of having complete access to all game parameters and cost functions of players for the static and dynamic cases. Subsequently, in Section IV, the same problem will be studied under information constraints and limitations.

When is it feasible to design a game such that the NE point can be located by the system designer to a point or region with desirable properties? Consider, without loss of any generality, the point case and denote the target point as $\hat{\mathbf{x}}$. Then, the problem is to find an $\hat{\alpha}$ such that $\hat{\alpha} = \hat{\mathcal{T}}(\hat{\mathbf{x}})$, for any desirable feasible $\hat{\mathbf{x}}$. The following surprisingly simple result addresses this problem for a broad class of games.

Theorem III.1. *For games of class $\mathcal{G}2$ with the cost structure given in (3) and under complete information assumption, affine pricing of the form, $\alpha\, p(\cdot)$, is sufficient to locate the unique NE point of the game to any desirable feasible point, $\hat{\mathbf{x}} \in \Omega$, as long as*

$$\frac{\partial p_i(\hat{\mathbf{x}})}{\partial \mathbf{x}_i} \neq 0, \quad \forall i.$$

Proof. The proof immediately follows from the first order necessary optimality conditions of player cost optimization problems due to the convexity of the cost structure and uniqueness of NE.

$$\alpha_i \frac{\partial p_i(\hat{\mathbf{x}})}{\partial \mathbf{x}_i} - \frac{\partial U_i(\hat{\mathbf{x}})}{\partial \mathbf{x}_i} = 0 \Rightarrow \hat{\alpha}_i = \left[\frac{\partial p_i(\hat{\mathbf{x}})}{\partial \mathbf{x}_i}\right]^{-1} \frac{\partial U_i(\hat{\mathbf{x}})}{\partial \mathbf{x}_i} \quad \forall i.$$

for any feasible $\hat{\mathbf{x}}$. □

Remark III.2. Theorem III.1 can easily be extended to the case where users actions are on a multi-dimensional subspace if the users utility function is separable.

Notice that even a simple linear pricing function $p(x_i) = x_i$ satisfies the conditions of the theorem and is sufficient for NE optimization. In this case any $\hat{\mathbf{x}} \in \Omega$ is feasible. However, a symmetric pricing scheme, where $\alpha_i = \alpha_j \;\; \forall i, j$, is not sufficient in general. As other examples, for $p(x_i) = e^{x_i}$ any $\hat{\mathbf{x}}$ is feasible, while for $p(x_i) = x_i^2$ any $\hat{\mathbf{x}} \neq 0$ is feasible.

If a game admits multiple NE, e.g. games of class $\mathcal{G}1$, then reaching a single desirable point does not make much sense. Furthermore, the problem of locating all NE points to a desirable region can be rather complex. Such cases can be handled either by exploiting any special structure of the game due to its specific problem domain or using numerical methods.

Theorem III.1 clearly establishes that "loss of efficiency" or "price of anarchy" is not an inherent feature of a broad class of games with built-in pricing systems, but merely a misconception that stems from arbitrary choice of game parameters. If there is sufficient information, then any game of class $\mathcal{G}2$ can be designed through simple pricing mechanisms in such a way that any desirable criteria such as welfare maximization or QoS requirements are met at the unique NE solution. An immediate question is of course the lack of information which we will address in the next section.

IV. NE Design under Information Constraints

In many problem formulations, the system designer does not have full information about the system parameters such as user preferences or utility functions. Under such information constraints, the designer either deploys additional dynamic feedback mechanisms or requires side information from the system, depending on the specific design objectives. An example for the former case is a dynamic pricing scheme operating as an "outer feedback loop". If the objective is to achieve a social optimum (e.g. maximization of sum of user utilities) or satisfying some QoS constraints, then the designer often needs *accurate and honest* side information from users or the system. Given such side information, the task of the designer can be formulated as an optimization problem even if it is solved indirectly or in a distributed manner. Here, the NE optimization is assumed to be on a slower time scale than the actual game dynamics leading to a time-scale separation, and hence to a hierarchically structured problem. Assuming this time-scale separation for simplicity, initially only the pricing dynamics (slower dynamics) are considered, [21].

We now investigate design problems with accurate but limited information. To illustrate the underlying concepts, two example formulations are provided. In the first formulation, the objective is to locate the NE to a region that satisfies some feasibility and QoS constraints. In the second one, the objective is to make the NE coincide with a social optimum (maximizing sum of user utilities) for the special case of separable user utilities of the form $U_i(x_i)$. For both cases, we consider a class $\mathcal{G}2$ game with the following general cost structure

$$J_i(\alpha_i, \mathbf{x}) = \alpha_i p_i(\mathbf{x}) - U_i(\mathbf{x}). \tag{4}$$

A. QoS-based Objective

Consider a game with cost structure given in (4) and utility function

$$U_i(\mathbf{x}) = \beta_i \log(1 + s_i(\mathbf{x})), \qquad \text{where } s_i(\mathbf{x}) := \frac{h_i x_i}{\sum_{j\neq i} h_j x_j + \sigma^2}. \tag{5}$$

Here, s represents a simple signal-to-interference ratio (SIR) with $h > 0$ denoting gain parameters and σ^2 a noise term. The desired region for the NE of this game could be shaped by feasibility constraints such as positivity of user actions and an upper-bound on the sum of them, and/or some chosen minimum SIR levels (assuming these are chosen such that the region is not empty). A concrete example region $\hat{\Omega}$ can be defined as

$$\hat{\Omega} := \{\mathbf{x} \in \mathbb{R}^N : \mathbf{x}_i \geq 0,\ s_i(\mathbf{x}) \geq \bar{s}_i\ \forall i\}, \quad (6)$$

where $\bar{s}_i$ are user-specific minimum SIR levels. A detailed analysis of an example case is provided next. For a separate but similar example of this formulation we refer to [5].

*1) **Example**:* Consider a single-cell spread-spectrum wireless uplink power control system with M users [22]. Each user i decides on its own power level x_i and is associated with the cost function J_i as in (4). The pricing function $p_i(\mathbf{x})$ is chosen to be linear in x_i such that

$$J_i(\alpha_i, \mathbf{x}) = \alpha_i x_i - \beta_i \log(1 + s_i(\mathbf{x})),$$

where $s_i(\mathbf{x})$ is defined in (5). Under appropriate assumptions, the game is one of class $\mathcal{G}2$ and admits a unique inner NE solution, $\mathbf{x}^*$.

For notational convenience, we define the matrix

$$A := \begin{pmatrix} 1 & \frac{h_2}{Lh_1} & \frac{h_3}{Lh_1} & \cdots & \frac{h_M}{Lh_1} \\ \frac{h_1}{Lh_2} & 1 & \frac{h_3}{Lh_2} & \cdots & \frac{h_M}{Lh_2} \\ \frac{h_1}{Lh_3} & \frac{h_2}{Lh_3} & 1 & \cdots & \frac{h_M}{Lh_3} \\ \vdots & \vdots & & \ddots & \vdots \\ \frac{h_1}{Lh_M} & \frac{h_2}{Lh_M} & \cdots & \frac{h_{M-1}}{Lh_M} & 1 \end{pmatrix} \quad (7)$$

Then, the NE is the solution of

$$A\mathbf{x}^* = \mathbf{c},$$

where

$$\mathbf{c} := \left[\frac{\beta_1}{\alpha_1} - \frac{\sigma^2}{Lh_1}, \ldots, \frac{\beta_M}{\alpha_M} - \frac{\sigma^2}{Lh_M} \right].$$

The desired QoS region $\hat{\Omega}$ in (6) can be alternatively described in terms of received power levels at the base stations and in matrix form [23]:

$$\hat{\Omega} = \{\mathbf{x} \in \mathbb{R}^N : \mathbf{x}_i \geq 0\ \forall i,\ S\mathbf{x} \geq \mathbf{b}\},$$

where the matrix S is defined as

$$S := \begin{pmatrix} h_1 & -h_2\frac{\bar{s}_1}{L} & \cdots & \frac{-h_M\bar{s}_1}{L} \\ \frac{-h_1\bar{s}_2}{L} & h_2 & \cdots & \frac{-h_M\bar{s}_2}{L} \\ \vdots & & \ddots & \vdots \\ \frac{-h_1\bar{s}_M}{L} & \frac{-h_2\bar{s}_M}{L} & \cdots & h_M \end{pmatrix}, \quad (8)$$

and

$$\mathbf{b} := \left[\frac{\bar{s}_1\sigma^2}{L}, \ldots, \frac{\bar{s}_M\sigma^2}{L} \right]^T.$$

If the designer, here the base station, has full information, then given a feasible target SIR level $\bar{s}$ it is possible to solve for a pricing vector α such that the NE is on the boundary of $\hat{\Omega}$, i.e. $S\mathbf{x} = \mathbf{b}$. This is due to both matrices A and S being nonsingular as $h_i > 0\ \forall i$. Hence, the appropriate pricing vector α can be immediately obtained from the boundary solution

$$\mathbf{c} = AS^{-1}(\mathbf{b}),$$

and the definition of $\mathbf{c}$.

However, in the limited information case where the designer does not have access to user preferences, a dynamic a pricing mechanism can be deployed. Toward this end, define a set of penalty functions $\rho_i(x_i)$ to bring the system within the desired region

$$\rho_i(\mathbf{x}_i) := \begin{cases} f(\mathbf{b}_i - (S\mathbf{x})_i), & \text{if } s_i < \bar{s}_i \\ 0, & \text{else} \end{cases}, \quad (9)$$

where the scalar function $f(\cdot)$ is smooth and nondecreasing in its argument, and $f(0) = 0$. For example, f could be a quadratic function.

A possible pricing function is then

$$\dot{\alpha}_i = \sum_j \frac{\partial \rho_j}{\partial x_j^*} \frac{\partial x_j^*}{\partial \alpha_i}\ \ \forall i. \quad (10)$$

It is assumed here that the designer (base station) has access to system parameters L, h, and σ. Therefore, the terms $\partial \rho_j / \partial x_j^*\ \forall j$ can be computed without any additional information. In addition, the designer can estimate the terms $\partial x_j^* / \partial \alpha_i\ \forall i$ through iterative observations [24].

Finally, this pricing mechanism ensures that the NE point of the underlying game, $\mathbf{x}^*$, enters the desired QoS region $\hat{\Omega}$. To show this, define the Lyapunov function

$$V := -\sum_i \rho_i(x_i)$$

on the compact game domain Ω. Taking the derivative of V with respect to time along the pricing dynamics (10) yields

$$\begin{aligned} \dot{V} &= -\textstyle\sum_i \frac{\partial \rho_i}{\partial x_i} \sum_j \frac{\partial x_i}{\partial \alpha_j} \dot{\alpha}_j \\ &= -\textstyle\sum_j \left(\sum_i \frac{\partial \rho_i}{\partial x_i} \frac{\partial x_i}{\partial \alpha_j} \right) \dot{\alpha}_j \\ &= -\textstyle\sum_j (\dot{\alpha}_j)^2 \leq 0, \end{aligned}$$

with $\dot{V} < 0$ outside the set $\hat{\Omega}$ and $\dot{V} = 0$ if and only if $\dot{\alpha}_i = 0\ \forall i$. Hence, the system converges to the desired region $\hat{\Omega}$ under the pricing mechanism.

B. Utility Maximization

Define a strictly concave and smooth social welfare function $\mathcal{U}(\mathbf{x})$ which is a sum of concave and *separable* utility functions $\mathcal{U}(\mathbf{x}) := \sum_i U_i(x_i)$ and admits a global maximum $\hat{\mathbf{x}} = \arg\max_{\mathbf{x}} \sum_i U_i(\mathbf{x}_i)$. This objective function constitutes a special case due to separability of user utilities and allows for design of a pricing scheme that brings the NE to the social maximum point *without necessarily requiring any side information*. As an example, consider a game with the following cost function $J_i(\alpha_i, \mathbf{x}) = \alpha_i p_i(\mathbf{x}) - U_i(x_i)$. Then, the social maximum is defined easily via the first order optimality conditions

$$\frac{\partial \mathcal{U}}{\partial \mathbf{x}}(\hat{\mathbf{x}}) = \left[\begin{array}{ccc} \frac{\partial \mathcal{U}}{\partial \mathbf{x}_1}(\hat{\mathbf{x}}) & \ldots & \frac{\partial \mathcal{U}}{\partial \mathbf{x}_N}(\hat{\mathbf{x}}) \end{array} \right] = 0.$$

Since $\mathcal{U}(\mathbf{x})$ is separable, the first order optimality conditions are $\dfrac{\partial U_i}{\partial x_i}(\hat{\mathbf{x}}_i) = 0 \ \ \forall i$.

We show that the social maximum coincides with the unique equilibrium (and NE) point of the following pricing mechanism

$$\dot{\alpha}_i = \sum_j \frac{\partial U_j}{\partial x_j^*} \frac{\partial x_j^*}{\partial \alpha_i} \ \ \forall i.$$

If these pricing dynamics are on a slower time scale than the game dynamics, then the system designer can obtain sufficiently accurate estimates of $\partial U_i(x_i^*)/\partial x_i$ and $\partial x_i^*/\partial \alpha_i$.

As one possibility, if the users adopt a gradient algorithm to solve the game, e.g. $\dot{x}_i = -\partial J_i/\partial x_i$, then the designer can use past values of $\mathbf{x}^*$ and α along with the exact form of the pricing functions p in (4) to estimate $\partial U_i(x_i^*)/\partial x_i$ directly without requiring any side information (except from some fixed system parameters) [24]. Another option is the users submitting their individual $U_i(x_i)$ values (but not the functions) to the designer with sufficient frequency to facilitate an accurate estimation. The full system, composed of pricing dynamics on the slow time scale (reduced system) and user dynamics on the fast time scale, can be analyzed by using a boundary layer approach as in [21]. For simplicity, we now focus only on the pricing dynamics (slow or reduced system).

Assume an ideal case where the parameter estimation is perfectly accurate. Then, the pricing mechanism above ensures that the NE point of the underlying game globally asymptotically converges to the maximum of the social welfare function.[2] The next theorem summarizes this result for the separable utility case and follows from Lyapunov theory and LaSalle's theorem in a straightforward manner by choosing the negative of social welfare function itself $\mathcal{U}$ as a Lyapunov function for the system.

Theorem IV.1. *Define an objective function* $\mathcal{U}(\mathbf{x}) := \sum_i U_i(x_i)$ *which admits a unique inner global maximum* $\hat{\mathbf{x}} = \arg\max_{\mathbf{x}} \mathcal{U}(\mathbf{x})$ *under suitable assumptions for user utilities* $U_i \ \forall i$ *in a class* $\mathcal{G}2$ *game. Further define the pricing mechanism*

$$\dot{\alpha}_i = \sum_j \frac{\partial U_j}{\partial x_j^*} \frac{\partial x_j^*}{\partial \alpha_i} \ \ \forall i, \tag{11}$$

Then, this pricing mechanism ensures that the NE point of the underlying game, $\mathbf{x}^*$*, globally asymptotically converges to the maximum of the social welfare function,* $\hat{\mathbf{x}}$*, if the Jacobian matrix of the mapping* $\mathcal{T}$ *with respect to the pricing vector* α*, defined as*

$$H(\alpha) := \frac{\partial \mathbf{x}^*}{\partial \alpha}(\alpha) = \left[\frac{\partial x_i^*}{\partial \alpha_j}(\alpha) \right], \ \ i,j = 1, \ldots, N,$$

is non-singular.

Proof. We analyze only the pricing dynamics (slow), assuming that the user dynamics is fast and converges quickly to $\mathbf{x}^* = \mathcal{T}(\alpha)$ for any given α. The pricing scheme (11) admits a unique equilibrium *if and only if* $\partial U_i/\partial x_i = 0 \ \forall i$. [3] The sufficiency statement immediately follows from (11).

To show necessity, for the NE $\mathbf{x}^*$ denote by

$$H(\alpha) = \frac{\partial \mathbf{x}^*}{\partial \alpha}(\alpha) = \left[\frac{\partial x_i^*}{\partial \alpha_j}(\alpha) \right], \ \ i,j = 1, \ldots, N,$$

the Jacobian matrix of the mapping $\mathcal{T}$ with respect to the pricing vector α. Then, using separability of the cost, (11) can be written in vector form as

$$\dot{\alpha} = H^T(\alpha) \ \left[\frac{\partial \mathcal{U}}{\partial \mathbf{x}}(\mathbf{x}) \right]^T. \tag{12}$$

Under the assumption that H is non-singular, it follows immediately that at the equilibrium point it is necessary that $\partial \mathcal{U}/\partial \mathbf{x} = 0$, which at the same time characterizes $\hat{\mathbf{x}}$. Consequently, at the unique equilibrium point of the pricing scheme the objective function $\mathcal{U}(\mathbf{x})$ reaches its maximum.

In order to establish convergence of (11), define a Lyapunov function similar to the one in *Example 1*:

$$V := -\sum_i U_i(x_i(\alpha))$$

on the compact game domain Ω and $\alpha \in \mathbb{R}^N$. Taking the derivative of V with respect to time along the pricing dynamics (11) yields

$$\begin{aligned} \dot{V} &= -\sum_i \frac{\partial U_i}{\partial x_i} \sum_j \frac{\partial x_i}{\partial \alpha_j} \dot{\alpha}_j \\ &= -\sum_j \left(\sum_i \frac{\partial U_i}{\partial x_i} \frac{\partial x_i}{\partial \alpha_j} \right) \dot{\alpha}_j \\ &= -\sum_j (\dot{\alpha}_j)^2 \leq 0. \end{aligned}$$

Thus $\dot{V} = 0$ only at $\dot{\alpha}_j = 0, \forall j$, or at its unique equilibrium. Hence, by the LaSalle's theorem, (Theorem 4.4, [21]), the pricing scheme (11) globally asymptotically converges to its unique equilibrium at which the NE solution coincides with the social maximum. □

*1) **Example**:* Consider a game with separable utility functions with the cost

$$J_i(\alpha_i, \mathbf{x}) = \alpha_i \left(\sum_i x_i \right) - U_i(x_i),$$

$$\text{where } U_i := \beta_i \log(1 + x_i) - k_i x_i).$$

This type of utility function may arise due to inherent physical constraints on player actions such as battery constraints on uplink transmission power levels in wireless devices. Then, the NE solutions is

$$x_i^* = \frac{\beta_i}{\alpha_i + k_i} - 1.$$

[2]For simplicity, the social maximum point is implicitly assumed to be on the solution space of the game.

[3]We drop for the rest of the proof the $(\cdot)^*$ notation characterizing the NE for convenience.

Notice that, the matrix $H(\alpha)$ is diagonal in this case. Furthermore, we can explicitly find

$$\frac{\partial x_i^*}{\partial \alpha_i} = -\frac{\beta_i}{(\alpha_i + k_i)^2} < 0,$$

from which non-singularity of H immediately follows. The properties of this example also hold for a quadratic pricing function replacing the linear one, i.e., $p_i(\mathbf{x}) = \sum_i x_i^2$. However, for $p_i(\mathbf{x}) = e^{\sum_i x_i}$, x_i^* is not independent of α_j and non-singularity of $H(\alpha)$ is not immediate.

V. Dynamic Control of NE

In many games that are solved by players in a distributed manner, convergence of the system trajectory to the NE may not be very fast, and hence the time-scale separation between system designers actions and actual game dynamics may fail. Then, the NE design can be modeled as a *feedback control system* which utilizes pricing as the control input and the desired target as the reference (see Figure 3). This formulation also brings a certain robustness with respect to initial conditions or game (system) parameters. The latter case is especially relevant for systems that are non-stationary over longer time periods and can also be formulated as a tracking problem. We refer to congestion and power control game formulations as specific examples [1], [13], [20].

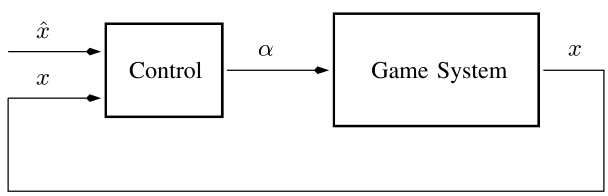

Fig. 3. Feedback control of the game (NE, $\mathbf{x}^*$) using pricing α as the control parameter and $\hat{\mathbf{x}}$ as the desired reference signal.

The counterpart of the feasibility question in the case of static NE optimization of the previous section in the dynamic control setting relates to the *controllability* of the system shown in Figure 3, or reachability of a state $\hat{\mathbf{x}}$. In order to provide a concrete example to the problem of controllability, consider a game of class $\mathcal{G}2$ where the players adopt a gradient algorithm to optimize their own cost. Then, the game dynamics are:

$$\dot{x}_i = -\frac{\partial J_i(\mathbf{x})}{\partial x_i} = \frac{\partial U_i(\mathbf{x})}{\partial \mathbf{x}_i} - \frac{\partial p_i(\mathbf{x})}{\partial \mathbf{x}_i}\,\alpha_i \;\; \forall i, \qquad (13)$$

where α acts as the feedback control on the outcome of the game. Here, the objective is to investigate the conditions under which the game system is controllable. We write (13) in vector form as

$$\dot{\mathbf{x}} = f(\mathbf{x}) + \sum_{i=1}^{N} g_i(\mathbf{x})\,\alpha_i = f(\mathbf{x}) + g(\mathbf{x})\,\alpha \qquad (14)$$

where $\alpha = [\alpha_1 \dots \alpha_N]^T$, $g(\mathbf{x}) = [g_1(\mathbf{x}), \dots, g_N(\mathbf{x})]$ and

$$f(\mathbf{x}) = \begin{bmatrix} \frac{\partial U_1(\mathbf{x})}{\partial \mathbf{x}_1} & \dots & \frac{\partial U_i(\mathbf{x})}{\partial \mathbf{x}_i} & \dots & \frac{\partial U_N(\mathbf{x})}{\partial \mathbf{x}_N} \end{bmatrix}^T$$

$$g(\mathbf{x}) = \begin{bmatrix} -\frac{\partial p_1(\mathbf{x})}{\partial \mathbf{x}_1} & \dots & 0 \\ \dots & -\frac{\partial p_i(\mathbf{x})}{\partial \mathbf{x}_i} & \dots \\ 0 & \dots & -\frac{\partial p_N(\mathbf{x})}{\partial \mathbf{x}_N} \end{bmatrix}$$

Based on the standard theorem on controllability using Lie brackets [25, Chapter 1], we obtain the following result.

Theorem V.1. *For games of class $\mathcal{G}2$ with the cost structure given in (3) and game dynamics (13), or (14), a sufficient condition for local reachability around a point $\mathbf{x}_0$ is that the distribution $\mathcal{C}$ satisfies the rank condition at $\mathbf{x}_0$, $dim\mathcal{C}(\mathbf{x}_0) = N$ where*

$$\mathcal{C} = \left[g_1, \dots, g_N, [g_i, g_j], \dots, [f, g_i], \dots, ad_f^k g_i, \dots\right]$$

where $[f, g](\mathbf{x}) = \frac{\partial g(\mathbf{x})}{\partial \mathbf{x}} f(\mathbf{x}) - \frac{\partial f(\mathbf{x})}{\partial \mathbf{x}} g(\mathbf{x})$ is the Lie bracket of f and g, and $ad_f^k g$ denote higher order Lie brackets defined recursively by $ad_f^k g(\mathbf{x}) = [f, ad_f^{k-1} g](\mathbf{x})$.

Remark V.2. Notice that if the diagonal matrix $g(\mathbf{x}_0)$ has rank N, any $\hat{\mathbf{x}}$ locally around $\mathbf{x}_0$ is reachable in finite time under piecewise constant input functions, which is equivalent to the feasibility condition in Theorem III.1. In addition, for the simple linear pricing function $p(x_i) = x_i$ any $\hat{\mathbf{x}}$ is immediately reachable since $g(\hat{\mathbf{x}})$ is constant and invertible.

VI. Conclusion

The general problem of Nash equilibrium design is discussed from an optimization and control theoretic perspective. A fairly general class of games is investigated that have been applied to a variety of settings including network congestion control, wireless uplink power control, and optical power control. The NE design is studied for various knowledge assumptions (full versus limited information) and design objectives (QoS versus utility maximization). Conditions for pricing functions that allow locating the NE solutions to any desired point are derived. In addition, convergence of two example dynamic pricing schemes is shown under the time-scale separation assumption between the game and pricing dynamics.

Ongoing work includes an extension of Theorem IV.1 for a full analysis of both pricing (slow) and user (fast) dynamics by using a singular perturbation approach and a combined Lyapunov function. A future research direction is the application of NE design methods to specific problems such as power control in optical networks and spectrum allocation in wireless networks. An additional direction is the analysis of estimation methods under limited information and the effect of estimation errors on performance.

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