A System Performance Approach to OSNR Optimization in Optical Networks

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***Abstract*—This paper studies a constrained optical signal-to-noise ratio (OSNR) optimization problem in optical networks from the perspective of system performance. A system optimization problem is formulated with the objective of achieving an OSNR target for each channel while satisfying the total power constraint. In order to establish existence of a unique optimal solution, the conditions are derived, which can be used as a basis for an admission control scheme. The original problem is then converted to a relaxed system problem by using a barrier function and solved by a distributed iterative algorithm. Next, the system optimization framework developed is compared to the game theoretic one in [1]. The effects of parameters in both formulations are investigated to study efficiency of Nash equilibria in the OSNR game and pricing mechanisms affecting overall system performance. The theoretical analysis is supported by numerical simulations and experiments conducted on an optical fiber link.**

I. Introduction

Multi-channel optical communication systems are realized by wavelength division multiplexing (WDM), which consists of several sources multiplexed in wavelength domain and transmitted over the same optical fiber. Control of optical networks via an optimization-based approach arises in the context of evolution of optical communications from statically designed point-to-point links, to reconfigurable WDM networks. A reconfigurable optical network operates dynamically, with existing channels being continuously served while network reconfiguration (e.g., channel added/dropped) is being performed. Essential research topics include optimization of channel performance with general topologies and online reconfiguration [2].

At the physical transmission level, channel performance is directly determined by the bit-error rate (BER), which in turn, depends on OSNR, dispersion and nonlinear effects [3]. OSNR is considered as the dominant performance parameter in link optimization, with dispersion and nonlinearity being limited with proper link design [4]. In multi-channel optical communication systems, a signal over the same optical link can be regarded as an interfering noise for others, which leads to the OSNR degradation [5]. Regulating the input optical power per channel at *Source* (transmitter site, Tx) aims to achieve a satisfactory OSNR level at *Destination* (receiver site, Rx). Such a satisfactory OSNR level, or OSNR target of each channel is regarded as *OSNR constraint*. Particularly in optical networks, because all wavelength-multiplexed channels in a link share the optical fiber, the total input power on a link has to be below the nonlinearity threshold [3], which can be regarded as the *total power constraint*. In order to avoid the nonlinearity effects of optical fiber, both the peak power of each wavelength and the total power of all wavelengths are important. In the multi-channel case, especially with a large number of channels, the total power constraint can be stronger than the individual channel power constraint. In this paper we study OSNR optimization in optical links from the perspective of system performance. The objective is to achieve an OSNR target for each channel while satisfying the total power constraint.

The OSNR optimization problem in optical networks belongs to a subclass of resource allocation in general communication networks [6]–[11]. The problem in optical networks is more challenging: a complex mathematical OSNR model due to cascaded optical amplified spans with a typical automatic power control (APC) operation mode and self-generation and accumulation of optical amplified spontaneous emission (ASE) noise in optical amplifiers, as well as specific constraints imposed by dispersion and nonlinear effects. A typical approach for OSNR optimization uses a static budget of impairments along an amplified fiber link with certain tolerance margins added such that at least a desired OSNR value is achieved. For example in [12], a simple, heuristic algorithm was devised for online OSNR equalization in long WDM links. Such a method requires no new equipment or adjustments at intermediate sites along the link. However its convergence was not considered.

The OSNR optimization problem in optical networks has been studied via a central cost approach in [5], with the objective of maintaining a desired OSNR for each channel while minimizing total channel input optical power. In this paper, the OSNR optimization problem is subject to more physical constraints, for example, the total power constraint is considered. Moreover, the system cost function of this optimization problem is more general than the one in [5]. The devised algorithm can be implemented in a distributed way with no new equipments added and unlike [12], its convergence is guaranteed.

The OSNR optimization problem was also studied in [1] via a noncooperative (Nash) game-theoretic approach [13]. In the Nash game, each channel is a player and competes to optimize its OSNR independently, i.e., to minimize its individual cost function by regulating its optical power at Tx, given the optical powers of all other channels at Tx. A solution to the Nash game is called a Nash equilibrium (NE) solution.

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Moreover, the Nash game immediately leads to distributed iterative algorithms towards finding the NE solution. Although in the game-theoretic framework, the total power constraint has been considered, the channel OSNR constraint was not. [1] provided an analytical approach to achieve given OSNR targets by tuning the parameters in associated game cost functions, which is also related to pricing mechanism [6].

It is well known that the Nash equilibria of a game may not achieve full efficiency due to the selfish behavior of players [6], [14]–[18]. The resulting degree of efficiency loss is known as the“price of anarchy” [19]. In particular, results have suggested that the system performance may not be degraded arbitrarily, provided that a pricing mechanism is chosen properly [6], [16]. A short version of this work appeared in [20] where only the system optimization setup was considered. In this paper, we are also interested to know how much cost is added or utility is lost due to the player's selfish behavior in a Nash game, based on the rigorous system optimization setup. Or in other words we consider the social welfare. There are many possible social welfare functions, one of which is the aggregate function. We use the system cost function as the social welfare function and we study the game efficiency by investigating the effects of parameters in individual game cost functions presented in [1]. We show that the aggregate cost function in the game theoretical formulation is not automatically strictly convex and finding the optimal solution of the associated constrained optimization problem is not immediate. We indicate that the system optimization problem we considered leads to an individual cost function that has an approximate interpretation as the game cost function in [1]. We compare the numerical results and show the effects of pricing mechanisms on the equilibria attained.

This paper is organized as follows. In Section II, we give some background of this work. A link OSNR model is reviewed and related work is presented. In Section refsec-sysoptiprob, we formulate the system optimization problem. In Section IV, a distributed iterative algorithm is proposed. We provide simulation and experimental results in Section V and we study the effects of parameters in the Nash game numerically in Section VI. Section VII gives conclusions and directions for future work.

II. Background

A. Link OSNR Model

Consider a point-to-point optical link shown in Fig. 1. A set $\mathcal{M} = \{1, \ldots, m\}$ of channels, corresponding to a set of wavelengths, are transmitted over the link from Txs to corresponding Rxs by intensity modulation and wavelength-multiplexing [3]. The link consists of N cascaded spans of optical fiber each followed by an optical amplifier (OA). It is assumed that all spans have equal length.

Channel optical power is attenuated during propagation through the optical fiber. OAs are deployed along optical links to amplify optical power of all channels simultaneously. All OAs are assumed to be operated in APC mode and have the same gain spectral shape and the gain value for channel i is G_i. Amplification is accompanied by the introduction of

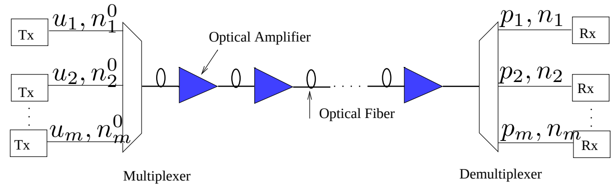

Fig. 1. Point-to-point optical link

ASE noise [3]. ASE noise is wavelength-dependent and accumulates along OAs during propagation. We denote $ASE_{s,i}$ the ASE noise power of channel i after span s defined as $ASE_i = 2n_{sp}(G_i - 1)h\nu_i B$, with n_{sp} the amplifier excess noise factor, h the Planck's constant, B the optical bandwidth and ν_i the optical frequency of channel i. Note that in the APC operation mode, the same total power is launched into each span of a link. The total power, denoted by P^0, is typically selected to be below a threshold to reduce nonlinear effects across optical fiber [3], [21].

We denote u_i and n_i^0 the signal optical power and noise optical power of channel $i \in \mathcal{M}$ at Tx, respectively. Similarly, we denote p_i and n_i the signal optical power and noise optical power of channel $i \in \mathcal{M}$ at Rx, respectively. Let $u = [u_1, \ldots, u_m]^T$ denote the vector form. Equivalently, we write (u_{-i}, u_i) with $u_{-i} = [u_1, \ldots, u_{i-1}, u_{i+1}, \ldots, u_m]^T$ in some context to represent the same vector u. The signal optical power at Tx is typically bounded for each channel. That is, u_i is in a bounded set $\Omega_i = [0, u_{max}]$ where u_{max} is sufficiently large. The bounded set Ω_i is assumed to be identical for all channels. We use $\Omega = \Omega_1 \times \cdots \times \Omega_m$ to represent its Cartesian product.

The OSNR of channel i at Rx is defined as $OSNR_i = \frac{p_i}{n_i}$. The following lemma [5] gives the OSNR model of a point-to-point optical link.

Lemma 1: The channel OSNR at Rx is given as

$$OSNR_i = \frac{u_i}{n_i^0 + \sum_{j \in \mathcal{M}} \Gamma_{i,j} u_j} \tag{1}$$

where $\Gamma = [\Gamma_{i,j}]$ is the system matrix with

$$\Gamma_{i,j} = \sum_{s=1}^{N} \frac{(G_j)^s}{(G_i)^s} \frac{ASE_{s,i}}{P^0} \tag{2}$$

Equivalently, (1) can be rewritten as $OSNR_i = \frac{u_i}{X_{-i} + \Gamma_{i,i} u_i}$, where $X_{-i} = n_i^0 + \sum_{j \in \mathcal{M}, j \neq i} \Gamma_{i,j} u_j$.

We study such an OSNR optimization problem in which the OSNR set of all channels is lower-bounded by a set of OSNR targets. Regulating the optical powers at Tx, i.e., allocating optical resource among channels, aims to achieve a satisfied OSNR level for each channel at Rx. We let $\widehat{\gamma}_i$ be channel i's target OSNR and the corresponding vector form is $\widehat{\gamma} = [\widehat{\gamma}_1, \ \ldots, \ \widehat{\gamma}_m]^T$. Thus the OSNR optimization problem is subject to the following *OSNR constraint*:

$$OSNR_i \geq \widehat{\gamma}_i, \ \forall \ i \in \mathcal{M} \tag{3}$$

Recall that when OAs are operated in the APC mode, the same total power P^0 is launched into each span of the link. This leads to a uniform total power distribution along the link, which minimizes nonlinear effects after each span [21]. However, this is not applicable to limit nonlinear effects on the optical fiber at the beginning of the link, i.e., the first segment of optical fiber. Thus the following condition (*total power constraint*) is considered to limit nonlinear effects: $\sum_{j\in\mathcal{M}} u_j \leq P^0$. In the OSNR optimization problem, it is imposed when regulating the optical powers at Tx.

B. Related Work

The OSNR optimization problem with the total power constraint on a single point-to-point optical link was formulated as a Nash game in [1].

In the game framework, the action space $\bar{\Omega} = \{u \in \Omega \,|\, \sum_{j\in\mathcal{M}} u_j - P^0 \leq 0\}$ is coupled. The action set for each channel i is the projection set $\hat{\Omega}_i(u_{-i}) = \{\xi \in \Omega_i \,|\, \sum_{j\in\mathcal{M}, j\neq i} u_j + \xi - P^0 \leq 0\}$. An individual cost function J_i assigned to each channel $i \in \mathcal{M}$ is defined as the difference between a *pricing function* P_i and a *utility function* U_i: $J_i = P_i - U_i$. The utility function U_i is chosen to be a logarithmic function of the associated channel's OSNR:

$$U_i = \beta_i \ln\left(1 + \frac{a_i}{1/OSNR_i - \Gamma_{i,i}}\right), \ \forall\, i \in \mathcal{M} \qquad (4)$$

where $\beta_i > 0$ is a channel-defined parameter indicating the strength of the channel's desire to maximize its OSNR and $a_i > 0$ is a scaling parameter for flexibility. From (4) we can see that U_i is monotonically increasing in $OSNR_i$. Hence, maximizing the utility function U_i is related to maximizing $OSNR_i$.

The pricing function consists of two terms: a linear pricing term and a regulation (penalty) term,

$$P_i(u_{-i}, u_i) = \alpha_i u_i + \frac{1}{P^0 - \sum_{j\in\mathcal{M}} u_j}, \ \forall\, i \in \mathcal{M}, \qquad (5)$$

where $\alpha_i > 0$ is a pricing factor determined by the system.

The linear pricing term can be interpreted as the price a channel pays for using the system resources. From the system performance point of view, the linear pricing term here is to limit the interferences of other channels caused by this channel and hence it improves the overall system performance [6].

The regulation term is constructed by considering the total power constraint. It indeed has a very fast rate of variations and high value when the total power $\sum_{j\in\mathcal{M}} u_j$ approaches to the total power constraint P^0. It has been shown in [1] that such a term affects the development of iterative algorithms. It also has been shown in [22] that the regulation term limits the maximum value of the step-size in the proposed gradient algorithm. However, such a regulation term directly consider the total power constraint and indirectly helps limit the interferences by forcing this channel to decrease its input power. It penalizes any violation of the constraint in the following way: The regulation term tends to infinity when the total power approaches the total power target P^0, so the pricing function $P_i(u_{-i}, u_i)$ increases without bound. Hence the system resource is preserved by forcing all channels to decrease their input powers and indirectly satisfies the total power constraint [23].

Thus the m-player Nash game is defined in terms of the cost functions $J_i(u_{-i}, u_i)$, $i \in \mathcal{M}$, played within the action space $\bar{\Omega}$. This game is called an *OSNR Nash game* and denoted by $GAME(\mathcal{M}, \hat{\Omega}_i, J_i)$. A vector u is called *feasible* if $u \in \bar{\Omega}$. The concept of an NE solution to this game is well defined [13]. In addition, the solution is called an *inner NE solution* if it is not on the boundary of the action space $\bar{\Omega}$. There exists an NE solution for this m-player OSNR Nash game and furthermore it is inner under sufficient conditions [1].

In $GAME(\mathcal{M}, \hat{\Omega}_i, J_i)$, the link sets fixed channel prices α_i and channels decide their willingness β_i to obtain higher OSNR levels. For a given OSNR target $\widehat{\gamma}_i$ for each channel i, i.e, a lower OSNR bound, it has been shown in [1] that if β_i satisfies

$$\beta_i > \frac{\alpha_i}{a_i} \frac{1 + (a_i - \Gamma_{i,i})\widehat{\gamma}_i}{1 - \Gamma_{i,i}\widehat{\gamma}_i} X_{-i} + \frac{(1 - \Gamma_{i,i}\widehat{\gamma}_i)(1 + (a_i - \Gamma_{i,i})\widehat{\gamma}_i)}{a_i(P^0 - \sum\limits_{j\neq i} u_j - (P^0 - \sum\limits_{j\neq i} u_j \Gamma_{i,i} + X_{-i})\widehat{\gamma}_i)^2} X_{-i}, \qquad (6)$$

each channel achieves at least the $\widehat{\gamma}_i$ level, i.e., $OSNR_i > \widehat{\gamma}_i$. Thus this result briefly provides an idea of how to achieve given OSNR targets by tuning the parameters in associated cost functions in the game-theoretic framework. Intuitively, larger β_i possibly leads to increase the lower OSNR bound.

III. System Optimization Problem

A. Problem Formulation

In contrast to the game-theoretic approach, a system optimization problem is formulated:

$$\begin{aligned} \min_{u} \quad & C(u) \\ \text{subject to} \quad & u_i \in \Omega_i, \ \forall\ i \in \mathcal{M}, \\ & OSNR_i \geq \widehat{\gamma}_i, \ \forall\ i \in \mathcal{M}, \text{(OSNR constraint)} \\ & \textstyle\sum_{j\in\mathcal{M}} u_j \leq P^0, \text{(total power constraint)}. \end{aligned}$$

The system cost function $C(u)$ is the sum of all individual cost functions, $C_i(u_i)$, $C(u) = \sum_{i\in\mathcal{M}} C_i(u_i)$. Each individual cost function $C_i(u_i)$ is generic and satisfies the following assumption:

Assumption 1: Each $C_i(u_i)$ is strictly convex, continuously differentiable and $\lim_{u_i\to 0} C_i(u_i) = +\infty$.

$C_i(u_i)$ can also be defined as the difference between a pricing term and a utility term. For example,

$$C_i(u_i) = \alpha_i u_i - \beta_i \ln u_i, \qquad (7)$$

where $\alpha_i > 0$ and $\beta_i > 0$.

By using the OSNR model in (1), we can rewrite the OSNR constraint as $u_i + \sum_{j\in\mathcal{M}} (-\widehat{\gamma}_i\, \Gamma_{i,j})\, u_j \geq n_i^0\, \widehat{\gamma}_i$. The associated vector form is $T\, u \geq b$, where

$$T = \begin{bmatrix} 1-\widehat{\gamma}_1\Gamma_{1,1} & -\widehat{\gamma}_1\Gamma_{1,2} & \cdots & -\widehat{\gamma}_1\Gamma_{1,m} \\ -\widehat{\gamma}_2\Gamma_{2,1} & 1-\widehat{\gamma}_2\Gamma_{2,2} & \cdots & -\widehat{\gamma}_2\Gamma_{2,m} \\ \vdots & \vdots & \ddots & \vdots \\ -\widehat{\gamma}_m\Gamma_{m,1} & -\widehat{\gamma}_m\Gamma_{m,2} & \cdots & 1-\widehat{\gamma}_m\Gamma_{m,m} \end{bmatrix}, \quad b = \begin{bmatrix} n_1^0\widehat{\gamma}_1 \\ n_2^0\widehat{\gamma}_2 \\ \vdots \\ n_m^0\widehat{\gamma}_m \end{bmatrix}. \qquad (8)$$

All off-diagonal entries of T, $-\widehat{\gamma}_i\, \Gamma_{i,j}$, are less than zero such that T is a Z-matrix [24].

From the total power constraint and $u_i \geq 0,\ \forall\ i \in \mathcal{M}$, we have $u_i \leq P^0$. Recalling that u_i is bounded in $\Omega_i = [0, u_{max}]$ and u_{max} is sufficiently large, we can deduce that the conditions $\sum_{j\in\mathcal{M}} u_j \leq P^0$ and $u_i \geq 0,\ \forall\ i \in \mathcal{M}$ are equivalent to $\mathbf{1}^T\ u \leq P^0$ and $u \geq 0$, where $\mathbf{1}$ is the $m \times 1$ all ones vector. Therefore, the constraint set of the system optimization problem is

$$\widetilde{\Omega} := \{u \in \mathbb{R}^m \,|\, T\,u \geq b,\ \mathbf{1}^T\,u \leq P^0 \ \text{ and } u \geq 0\}.$$

The constrained system optimization problem is formulated as $\min_{u\in\widetilde{\Omega}} C(u)$. We denote the system optimization problem by $OPT(\widetilde{\Omega}, C)$. The condition in Assumption 1 ensures that the solution to $OPT(\widetilde{\Omega}, C)$ does not hit $u_i = 0,\ \forall\, i \in \mathcal{M}$. The following result characterizes the unique solution of $OPT(\widetilde{\Omega}, C)$.

Theorem 1: If the following conditions on $\widehat{\gamma}$ hold:

$$\widehat{\gamma}_i < \frac{1}{\sum_{j\in\mathcal{M}} \Gamma_{i,j}},\ \forall\, i \in \mathcal{M}, \tag{9}$$

where $\Gamma = [\Gamma_{i,j}]$ is the system matrix defined in (2) and

$$\mathbf{1}^T \cdot \widetilde{T}(\widehat{\gamma}) \cdot b(\widehat{\gamma}) \leq P^0, \tag{10}$$

with $b(\widehat{\gamma}) = [n_1^0\widehat{\gamma}_1, \ldots, n_m^0\widehat{\gamma}_m]^T$ and $\widetilde{T}(\widehat{\gamma}) = T^{-1}(\widehat{\gamma})$, then the constraint set $\widetilde{\Omega}$ is non-empty and $OPT(\widetilde{\Omega}, C)$ has a unique positive solution u^{opt}.

The conditions in Theorem 1 are tight enough. They are sufficient conditions to make the constraint set non-empty and to ensure that the optimization problem has a unique positive solution. Since these are not necessary, there might be some other bound attaining the same results.

Proof: We first show that the constraint set $\widetilde{\Omega}$ is non-empty. Note that in the link OSNR model, the system matrix Γ (2) is a positive matrix, so if (9) is satisfied, we have

$$1 - \widehat{\gamma}_i\Gamma_{i,i} > \widehat{\gamma}_i \sum_{j\in\mathcal{M}, j\neq i} \Gamma_{i,j} > 0,\ \forall\ i \in \mathcal{M},$$

which implies that the Z-matrix T has positive diagonal entries and T is strictly diagonally dominant [25]. According to Gershgorin's Theorem [25], each eigenvalue of T has a positive real part. Then it follows from Theorem 5.1 in [24] that T is an M-matrix and it has the following properties: $T\,u \geq b > 0$ implies $u \geq 0$ (for vector inequalities, we write $a \geq b$ if all $a_i \geq b_i$ and $a > b$ if all $a_i > b_i$), and T^{-1} is non-negative. Thus $u \geq T^{-1}\,b := \widetilde{T}\,b$ and then we have $\mathbf{1}^T\,u \geq \mathbf{1}^T \cdot \widetilde{T} \cdot b$. Note that both $\widetilde{T}$ and b depend on $\widehat{\gamma}$, i.e., $\widetilde{T} = \widetilde{T}(\widehat{\gamma})$ and $b = b(\widehat{\gamma})$. So $\mathbf{1}^T \cdot u \geq \mathbf{1}^T \cdot \widetilde{T}(\widehat{\gamma}) \cdot b(\widehat{\gamma})$. Let $\widetilde{u} = \widetilde{T}\,b$. Then $T\,\widetilde{u} = b$. Also we have $\mathbf{1}^T \cdot \widetilde{u} = \mathbf{1}^T \cdot \widetilde{T}\,b$. By (10), $\mathbf{1}^T \cdot \widetilde{u} \leq P^0$. It follows that $\widetilde{u} \in \{u \in \mathbb{R}^m \,|\, T\,u \geq b,\ \mathbf{1}^T\,u \leq P^0\}$. Thus the above set is non-empty if both (9) and (10) are satisfied. Since $T\,u \geq b > 0$ implies $u \geq 0$, we have proved that if $\widehat{\gamma}$ is selected such that (9) and (10) are satisfied, the constraint set $\widetilde{\Omega}$ is non-empty.

Moreover, the constraint set $\widetilde{\Omega}$ is convex and we have $0 \leq u_i \leq P_0, \forall i \in \mathcal{M}$. So $\widetilde{\Omega}$ is bounded. In addition, it is also closed since it consists of the intersection of half-spaces. Thus this system optimization problem is a strictly convex optimization problem on a convex compact constraint set, which always admits a unique globe minimum, u^{opt}. ■

Remark 1: Recall that n^0 denotes the input noise power at Tx and also can be considered to include some external noise, such as thermal noise. Then if the input noise is neglected, n^0 includes only some other external noise, which is also negligible [4]. So $b = [n_1^0\widehat{\gamma}_1, \ldots, n_m^0\widehat{\gamma}_m]^T \approx 0$ and therefore $P^0 \geq \mathbf{1}^T \cdot \widetilde{T} \cdot b \approx 0$, which means the constraint set is non-empty provided that the condition (9) is satisfied. Thus the OSNR target $\widehat{\gamma}_i$ can be selected in a distributed way based on the first condition (9).

B. Maximum OSNR target: $\widehat{\gamma}_{max}$

Let us take a close look at the condition (10). In a real network system, it is always a question how to express the conditions under certain physical constraints. Recall that $T = I - diag(\widehat{\gamma})\,\Gamma$, where $diag(u)$ is a diagonal matrix whose diagonal entries are elements of the vector u. We know from the proof of Theorem 1 that the matrix T is an M-matrix. By Theorem 5.1 in [24], $\rho(diag(\widehat{\gamma})\Gamma) < 1$ and

$$T^{-1} = (I - diag(\widehat{\gamma})\Gamma)^{-1} = \sum_{k=0}^{\infty} diag(\widehat{\gamma}^k)\Gamma^k$$

exists which is positive component-wise. We rewrite (10) as

$$\mathbf{1}^T \cdot \sum_{k=0}^{\infty} diag(\widehat{\gamma}^k)\Gamma^k \cdot diag(\widehat{\gamma}) \cdot n^0 \leq P^0. \tag{11}$$

If $\widehat{\gamma}_i$ increases (given $\widehat{\gamma}_j,\ j \neq i$), the left-hand-side (LHS) of (11) will increase. We can find a maximum OSNR target $\widehat{\gamma}_{max}$ by solving the following equation:

$$\widehat{\gamma}_{max} \cdot \mathbf{1}^T \cdot (I - \widehat{\gamma}_{max}\Gamma)^{-1} \cdot n^0 = P^0 \tag{12}$$

There is no direct relation between $\widehat{\gamma}_{max}$ and $C(u)$. However, the maximum OSNR target affects the solution of the optimization problem with the cost function $C(u)$.

Based on the link OSNR model, we know that the performance for each channel is interference-limited. In addition, (12) shows that the OSNR target levels significantly affect the capacity of an optical link: The link decides the OSNR threshold $\widehat{\gamma}_{max}$ by using (12). Though there is no direct relation between the maximum OSNR target $\hat{\gamma}_{max}$ and $C(u)$, $\widehat{\gamma}_{max}$ affects the constraint set over which the solution of the optimization problem with the cost function $C(u)$ is found. Any new channel with a required OSNR level no more than $\widehat{\gamma}_{max}$ will be admitted to transfer over the link (by using the sufficient conditions). With further necessary condition, this idea can be developed for links to devise channel admission control schemes.

IV. Distributed Algorithm

Recall that $OPT(\Omega, C)$ is a constrained optimization problem. There are several computational methods for solving the constrained optimization problem [26]. In this section, we use a barrier (or penalty) function to relax the constrained optimization problem. We first show that by appropriate choice of barrier functions, the solution of the relaxed system problem can arbitrarily approximate the one of the original problem

$OPT(\Omega, C)$. Then a distributed primal algorithm is presented for the relaxed system problem.

Theorem 1 shows that $T\,u \geq b > 0$ implies $u \geq 0$. Then $OPT(\Omega, C)$ can be rewritten succinctly as

$$\min_{u} \quad C(u) \qquad \text{subject to} \quad \widehat{T}\,u \geq \widehat{b}, \tag{13}$$

where $\widehat{T} = \begin{bmatrix} T \\ -\mathbf{1}^T \end{bmatrix}$ and $\widehat{b} = \begin{bmatrix} b \\ -P^0 \end{bmatrix}$.

A. Relaxed System Problem

A barrier function, $\lambda_i : \mathbb{R} \rightarrow \mathbb{R}$, is selected with the following properties:

(P.1) $\forall i \in \mathcal{M}$, $\lambda_i(x)$ is non-increasing, continuous and

$$\lim_{u_i \to \infty} \int_{\widehat{b}_i}^{y_i(u)} \lambda_i(x)dx \rightarrow -\infty \tag{14}$$

where

$$y_i(u) := row_i(\widehat{T})u. \tag{15}$$

Note that the barrier function λ_i should go to negative infinite when the constraint $y_i(u) < \widehat{b}_i$ is violated.

(P.2) $\lambda_i(x) = 0$ if $x > \widehat{b}_i$, where $\widehat{b}_i$ is defined in (13).

By using a barrier function λ_i with properties (P.1) and (P.2), we construct a function

$$V_p(u) = \sum_{i\in\mathcal{M}} C_i(u_i) - \sum_{i\in\mathcal{M}} \int_{\widehat{b}_i}^{y_i(u)} \lambda_i(x)dx. \tag{16}$$

Based on $V_p(u)$, we establish a relaxed system problem:

$$\min_{u\geq 0} V_p(u). \tag{17}$$

Recall that the cost function $C_i(u_i)$ is strictly convex and $C(u)$ is also strictly convex. Thus the non-increasing property of the barrier function together with Assumption 1 in Section III-A ensures that $V_p(u)$ is strictly convex [11]. Then it has a unique internal minimum value satisfying the following equations,

$$\frac{\partial V_p(u)}{\partial u_i} = C_i'(u_i) - row_i(\widehat{T}^T)\lambda(y(u)) = 0,\ \forall\, i \in \mathcal{M}.$$

Thus solving the set of above equations we obtain the unique solution $\bar{u}^{opt}$ of (17):

$$\bar{u}_i^{opt} = C_i^{'-1}(row_i(\widehat{T}^T)\lambda(y(\bar{u}^{opt}))),\ \forall\, i \in \mathcal{M}.$$

The barrier function $\lambda_i(\cdot)$ can be selected such that the unique solution of (17) may arbitrarily closely approximate the optimal solution of $OPT(\Omega, C)$. For example, a barrier function can be defined as $\lambda_i(x) = \frac{[\widehat{b}_i - x + \epsilon]^+}{\epsilon^2}$, where $[x]^+ = \max\{x, 0\}$. Thus as $\epsilon \rightarrow 0$, the relaxed system problem approximates the solution of the original system problem arbitrarily close [10].

B. Primal Algorithm

Now we develop a distributed algorithm for the relaxed system problem. A primal algorithm is defined as a set of following differential equations:

$$\dot{u}_i(t) = -k_i \frac{\partial V_p(u)}{\partial u_i} = -k_i \left(C_i'(u_i(t)) - s_i(t)\right), \tag{18}$$

where the coefficient $k_i > 0$ and $s_i(t)$ is defined as:

$$s_i(t) = row_i(\widehat{T}^T)\lambda(y(u(t))), \tag{19}$$

where $\lambda(\cdot) = [\lambda_1(\cdot),\ \ldots,\ \lambda_{m+1}(\cdot)]^T$ is a pre-defined barrier function vector.

The algorithm (18) is a gradient algorithm and can be implemented in a distributed way. Each channel varies its input power u_i gradually as in (18), while the link (network system) calculates the vector $s(t) = [s_1(t), \cdots, s_m(t)]^T$ based on the received input powers, OSNR preference and link constraint, and then feeds this updated information back to each channel.

The following theorem states that the unique equilibrium of the algorithm (18) corresponds to the unique solution of (17), $\bar{u}^{opt}$. Moreover the solution is globally asymptotically stable.

Theorem 2: The unique solution $\bar{u}^{opt}$ to the relaxed system optimization problem (17) is globally asymptotically stable for the system (18).

Proof: Notice that $\bar{u}^{opt}$ is the unique solution to the equations $\frac{\partial V_p(u)}{\partial u_i} = 0$, $\forall\ i \in \mathcal{M}$. Thus, it is the unique equilibrium point of the system (18).

Since $V_p(u)$ is strictly convex, it follows that $\bar{u}^{opt}$ is the global minimum point of the function $V_p(u)$. Let $C = V_p(\bar{u}^{opt})$. Then $V_p(u) > C$ for all $u \neq \bar{u}^{opt}$.

Now we construct a Lyapunov function for the system (18): $V(u) = V_p(u) - C$. Then it can be easily verified that $V(u) = 0$ when $u = \bar{u}^{opt}$, and that $V(u) > 0$ when $u \neq \bar{u}^{opt}$. That is, the function $V(u)$ is positive definite with respect to the equilibrium point $u = \bar{u}^{opt}$.

Taking the derivative of $V(u)$ along the trajectory of the system gives

$$\dot{V}(u) = \sum_{i\in\mathcal{M}} \left(\frac{\partial}{\partial u_i} V_p(u) \cdot \dot{u}_i\right) = -\sum_{i\in\mathcal{M}} k_i \left(\frac{\partial}{\partial u_i} V_p(u)\right)^2.$$

Thus we know that $\dot{V}(u) = 0$ when $u = \bar{u}^{opt}$, and that $\dot{V}(u) < 0$ when $u \neq \bar{u}^{opt}$. It means $\dot{V}(u)$ is negative definite with respect to the equilibrium point $u = \bar{u}^{opt}$. Hence, by Lyapunov stability theory the conclusion follows. ■

The unique solution of the the relaxed system problem (17) may arbitrarily closely approximate the optimal solution of the original system problem (13) with an approximate selection of the barrier function.

V. Simulation and Experimental Results

In this section, we present MATLAB simulation results and experimental results for a point-to-point optical link shown in Fig. 1 by using the algorithm (18).

A. Simulation Results

In simulation, the link has six channels ($m = 6$) and the link total power target is $P^0 = 2.5\,\text{mW}$ ($3.98\,\text{dBm}$). Within the set of six channels, there are two levels of OSNR target, a $\widehat{\gamma}_i = 26$ dB level desired on the first three channels, $i = 1, 2, 3$ and a $\widehat{\gamma}_i = 22$ dB OSNR level on the next three channels, $i = 4, 5, 6$. The conditions (9) and (10) on the target OSNR are satisfied. So the feasible constraint set is non-empty. The cost function for channel i is defined as in (7) with $\alpha_i = 1$, $i = 1, \cdots, 6$, and $\beta = [0.5,\ 0.51,\ 0.52,\ 0.3,\ 0.31,\ 0.32]$. Primal algorithm (18) is applied, where the coefficient k_i is fixed for each channel with $k_i = 0.01$, $i = 1, \cdots, 6$. We initially set the channel power as $u(0) = [0.216\ 0.221\ 0.226\ 0.231\ 0.236\ 0.833]$ (mW). The barrier function is selected as

$$\lambda_i(x_i) = 1000 \left(\max\{0, \hat{b}_i - x_i\} \right)^6, \qquad (20)$$

where $x_i(u) = row_i(\widehat{T})u$. Notice that $\lambda_i(u_i)$ is zero when the constraints are satisfied. So there is a penalty with any violation of the constraints. By using the primal algorithm to adjust all channel powers, two desired OSNR targets are achieved after the iterative process with the total power not exceeding the link total power constraint. The channel OSNR vs iteration time is shown in Fig. 2.

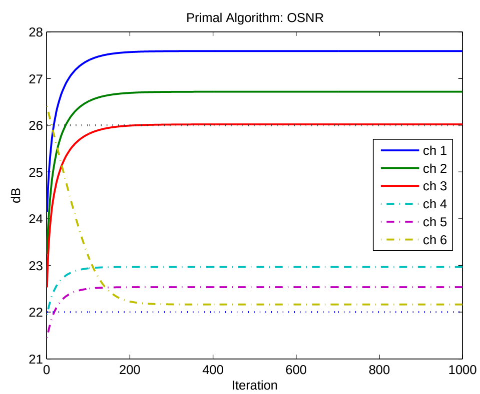

Fig. 2. Primal: OSNR

As a comparison, we provide some numerical results in order to compare performance of the primal algorithm and an OSNR equalization algorithm proposed in [12]. In the original OSNR equalization algorithm, each channel updates its input power at iteration time $(n+1)$ as

$$u_i(n+1) = P^0 \frac{u_i(n)/OSNR_i(n)}{\sum_{j \in \mathcal{M}} (u_j(n)/OSNR_j(n))}, \qquad (21)$$

with constant total power P^0 is maintained. The channel OSNR vs iteration time is shown in Fig. 3. Channel OSNR is equalized in a few iterations but the corresponding two desired OSNR targets are not satisfied. Note that the solution is non-optimal and convergence is not ensured theoretically.

B. Experimental Results

Experimental results are developed to test application of control and optimization theory to optical communication networks. The experimental setup has a number of devices used in the optical industry interconnected into a self-contained, small scale optical link emulation testbed that is representative of a real system while at the same time offering the flexibility

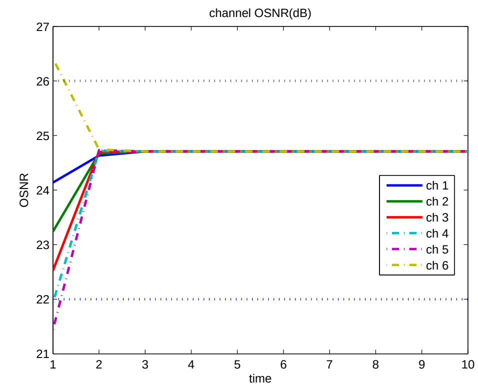

Fig. 3. OSNR equalization: Original

in monitoring and control in real time via optical spectrum analyzer and PC based control system.

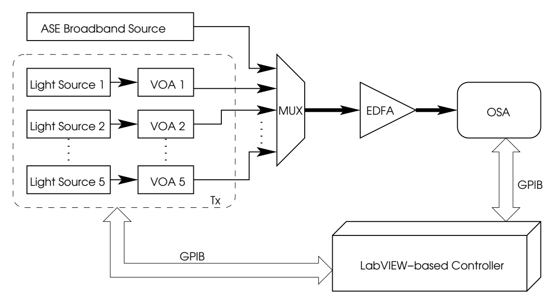

Fig. 4. Block-Diagram

We present experimental verification of the distributed primal algorithm (18) implemented in a LabVIEW setup shown in Fig. 4. Each transmitter (Tx) is composed of a CW (continuous-wave) light source connected to a variable optical attenuator (VOA), emitting at five ITU grid wavelengths $1533.47, 1535.04, 1537.40, 1555.75$ and 1558.17 (nm), respectively. Output power of each transmitter is adjusted by tuning the corresponding VOA. Multiple wavelengths, corresponding to multiple channels, are coupled together by using a multiplexer (MUX) and transmitted over an optical link with cascaded optical amplifiers (EDFA). The input noise is generated by an ASE broadband source. An optical spectrum analyzer (OSA) acts as the receiver (Rx) to measure the channel power and channel OSNR at the output of the link. Within the set of channels, there are two levels of OSNR target, a 28 dB level desired on the first channel and a 24 dB OSNR level on the other four channels. The optical link has a constant total power target $P^0 = 2.5$ mW.

In the proof of concept testbed a controller is used in the experiment to control powers of all channels (see Fig. 4). However, the primal control algorithm for each channel can be implemented in a decentralized way. Each channel varies its input power u_i gradually as in (18), while the link calculates the vector $s(t) = [s_1(t), \cdots, s_m(t)]^T$ based on the received input channel powers, OSNR preference and link constraint, and then broadcasts the updated information back to each channel. Specifically, in the experiment $C_i(u_i)$ is selected as in the numerical simulation section, (7), (20). In order to reduce the computational load on the link, instead of using (19) directly, the link calculates the $s_i(t)$ based on the measured channel OSNRs:

$$s_i(t) = \sum_{j=1}^{m} T_{j,i}\lambda_j \left(\frac{u_j(t)}{OSNR_j(t)} - n_j^0 \right) - \lambda_{m+1} \left(- \sum_{j=1}^{m} u_j(t) \right).$$

This update takes into account the actual link rather than the model. Another essential implementation issue is the calculation of the link system matrix, $\Gamma = [\Gamma_{i,j}]$, (2). Thus the link system matrix Γ for an optical link with EDFA is easy to calibrate. Moreover, it has been checked that in such a Γ-based link system, the conditions in Theorem 2 based on pre-defined OSNR constraints are satisfied.

Fig. 5 shows the snapshot of the front panel in LabVIEW after the control algorithm runs 100 iteration time. In total, the time taken for a basic control loop using these lab devices is approximately 60 seconds. We see that channel OSNR, channel output power and total channel power converge to stable states. Total channel power keeps below the total power target P^0. The final channel OSNRs are 26.93 dB, 26.90 dB, 25.28 dB, 26.02 dB and 28.71 dB, which satisfy the OSNR constraints. The experimental result validate the applicability of the achieved theoretical results.

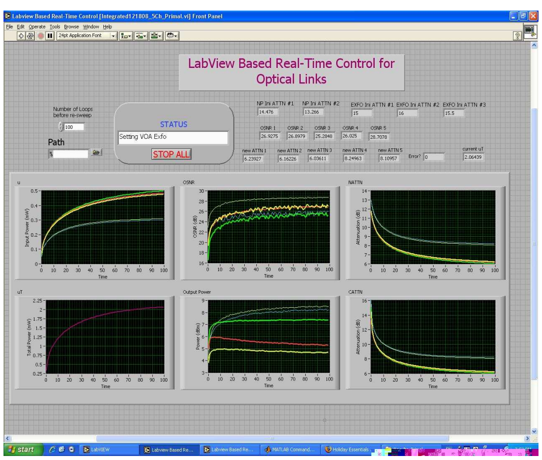

Fig. 5. LabView Snapshot of Results

VI. Parameter Effects in the Nash Game

We use the system optimization framework to investigate the effects of parameters in the game-theoretic framework.

A. Comparison of Cost Functions

In $GAME(\mathcal{M}, \hat{\Omega}_i, J_i)$, we are interested to know how the social welfare changes due to the player's selfish behavior in a Nash game. There are many possible social welfare functions, one of which is the aggregate function. However, generally the convexity of the aggregate game cost function defined as $J(u) := \sum_{i\in\mathcal{M}} J_i(u)$ is not longer guaranteed. The following simple example illuminates this, in which we omit the penalty term and noise in the OSNR model for simplicity.

Example 1: Consider a Nash game with 3 players and

$$J_i(u) = u_i - \ln \frac{u_i}{\sum_{j\neq i} u_j}, \ i = 1, 2, 3.$$

It follows that

$$\frac{\partial^2 J}{\partial u_1^2} = \frac{1}{u_1^2} - \frac{1}{(u_1+u_3)^2} - \frac{1}{(u_1+u_2)^2}$$

The sign of $\frac{\partial^2 J}{\partial u_1^2}$ is uncertain. Thus $J(u)$ is not always convex with respect to u_1. □

The constrained optimization problem associated with an aggregate cost function is not always a convex optimization problem and the optimal solutions are not immediate. Recall that an individual cost function $C_i(u_i)$ in $OPT(\Omega, C)$ has an approximate interpretation similar to the one of the cost function $J_i(u)$ in $GAME(\mathcal{M}, \hat{\Omega}_i, J_i)$. By this approximate definition, the individual cost function C_i is uncoupled in u for a given set of other power u_{-i}. Furthermore, it has an approximate interpretation similar to the one of J_i. Thus we build the relation between these two formulations, i.e., $OPT(\Omega, C)$ and $GAME(\mathcal{M}, \hat{\Omega}_i, J_i)$. We use the central cost function in $OPT(\Omega, C)$ approximately as the welfare function of $GAME(\mathcal{M}, \hat{\Omega}_i, J_i)$. We will compare simulation results based on this later. Moreover, in the next we select the system optimization framework to measure the efficiency of the NE solution numerically.

B. Parameter Effects in the Nash Game

As in Section V-A, the approximate optimal solution of $OPT(\Omega, C)$ is achieved as $u^{opt} = [0.5\ 0.51\ 0.52\ 0.3\ 0.31\ 0.32]$(mW). The system cost value is $C(u^{opt}) = 4.5789$. The NE solution of $GAME(\mathcal{M}, \hat{\Omega}_i, J_i)$ is denoted by u^*.

We first present three cases in which the parameter selection strategy is not used as a guideline (thus it is possible that the game settles down at an NE solution where channels do not reach their OSNR targets). In all cases, the user-defined parameters β_i in $GAME(\mathcal{M}, \hat{\Omega}_i, J_i)$ are chosen as same as β_i in $OPT(\Omega, C)$. The link sets fixed α_i at 0.001, 1 and 20, respectively. With these pricing mechanisms, the total power (u_T) is maintained in all the cases below the link constraint, but the channel OSNR target is achieved only for the some channels and only for α_i at 0.001 and 1.

Thus it is observed that without proper pricing mechanism, OSNR targets may not be achieved for some channels or all channels. The link capacity constraint is satisfied in all cases. With larger α_i (say, $\alpha_i = 20$ in the third case), total power is smaller than the link capacity constraint. While with smaller α_i in the first two cases, total power approaches the constraint and higher channel OSNR is possibly achieved.

Channel powers, u^{opt} and u^* in three games are shown in Fig. 6 vs channel number. The system cost $C(u)$ is evaluated by plugging in u^{opt} and u^*. Recall that $C(u^{opt}) = 4.5789$. For the game cases we get: $C(u^*) = 4.7403$ for $\alpha_i = 0.001$, $C(u^*) = 4.9282$ for $\alpha_i = 1$, and $C(u^*) = 9.7804$ for $\alpha_i = 20$. Results imply that larger α_i degrades system performance and even violates the system constraints.

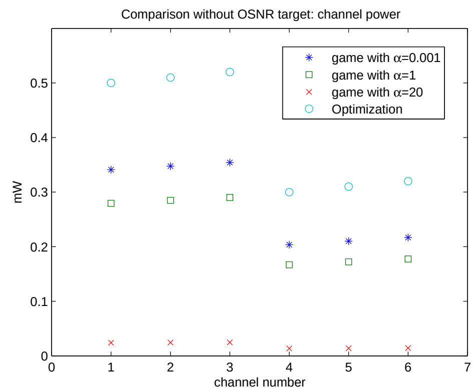

Fig. 6. Comparison: Channel power in games

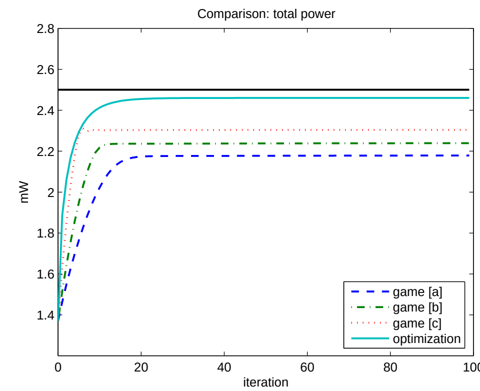

Fig. 7. Comparison: Total power

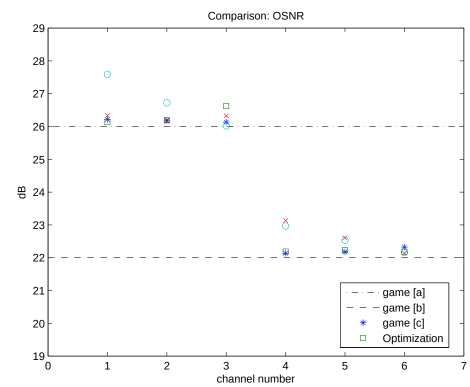

Fig. 8. Comparison: Channel OSNR

Next we present three other cases in which proper pricing mechanisms are chosen such that OSNR targets for all channels are achieved. We compare by simulation the two approaches: system optimization approach and the game theoretical approach. In $GAME(\mathcal{M}, \hat{\Omega}_i, J_i)$, the parameter selection strategy (6) is used such that proper pricing mechanisms are chosen and OSNR targets for all channels are achieved. Although the parameter selection strategy acts as a guideline for the selection of each β_i, it is rather intractable. Thus we choose proper pricing mechanism in simulation by trial and error. The parameters α_i are set at 1 for all cases and β_i are selected as in Table I such that different pricing mechanisms are chosen for $GAME(\mathcal{M}, \hat{\Omega}_i, J_i)$. Since we do not use Monte Carlo method [27] to simulate, we select β_i in three games by using the following rules. Firstly, β_i increases for each channel, i.e., Game [c] has the largest β_i compared to Game [a] and Game [b]. Secondly, Game [b] has the largest ratio of β_i to β_{min}.

TABLE I
PARAMETERS: β_i

	β_i
Game $[a]$	$[3.8\ 4.8\ 5.8\ 2.6\ 3.0\ 3.5]$
Game $[b]$	$[5.5\ 7.0\ 9.4\ 4.0\ 4.5\ 5.0]$
Game $[c]$	$[10\ 12\ 15\ 8.4\ 8.5\ 8.3]$

The efficiency of the corresponding solutions u^* is compared by evaluating the system cost $C(u)$ for u^* and comparing it to $C(u^{opt}) = 4.5789$. The corresponding system cost values are: $C(u^*) = 4.6171$ for Game [a], $C(u^*) = 4.6216$ for Game [b], and $C(u^*) = 4.6057$ for Game [c].

These results verify that the efficiency in the solution of the Nash game can be improved by proper pricing mechanism. The fact that no full efficiency in the solution of the Nash game is a well-known fact in the literature of economics [14], transportation [15] and network resource allocation [16]. Moreover, the Nash game solution gets very close to the optimal solution for system optimization. Furthermore, we can see that the NE solution in Game [c] is most efficient among these three cases. It implies that the efficiency can be possibly improved by appropriate selection of parameters.

Fig. 7 shows the total power vs iteration and channel OSNR vs channel number is shown Fig. 8. The constraints are satisfied in all cases. The total power in Game [c] approaches P^0 more than others. Moreover, we can tell from Fig. 8 that among the three cases, channel final OSNR values in Game [c] approach the optimal solution of $OPT(\Omega, C)$ most.

VII. CONCLUSION

We have studied a constrained OSNR optimization problem in optical networks from the perspective of system performance. As a first step, we have studied the single point-to-point link case. Each channel maintains a desired OSNR level. Meanwhile, it optimizes its input power regarding target OSNR levels of all other channels and link capacity constraint. Given reasonable target OSNR levels for all channels, the system optimization problem admits a unique solution. By using a barrier function, we have relaxed the original constraint system optimization problem into an unconstrained optimization problem and a distributed primal algorithm was developed. Extension and generalization of the results from the single link case to the network case is an interesting future research direction. Simulation and experimental results via the system optimization approach have been presented. Furthermore, we have used the system optimization framework to measure the efficiency of Nash equilibria of the Nash game. We have numerically investigated the effects of parameters in individual game cost functions. Simulation results have shown that OSNR target in the game-theoretic framework can be achieved and the efficiency can be possibly improved by appropriate selection of parameters. Future work will address this efficiency study theoretically, [28].

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